Toroidal compactifications of torsion free local complex hyperbolic spaces

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Abstract

Let $X' = (\mathbb{B}^n/\Gamma)'$ be the toroidal compactification of a torsion free local complex hyperbolic space $\mathbb{B}^n/\Gamma$. For an arbitrary Γ-rational boundary point $p \in \partial_\Gamma \mathbb{B}^n$, denote by $U(p)$ the commutant of the unipotent radical of $Stab(p) < SU_{n,1}$. The present note establishes that the fundamental group $\pi_1(X') \simeq \Gamma/\Gamma^U$ for the normal subgroup $\Gamma^U = \langle \Gamma \cap U(p) \quad | \quad \forall p \in \partial_\Gamma \mathbb{B}^n \rangle$ of Γ. As a consequence, $H_1(X', \mathbb{Z}) \simeq H_1(\mathbb{B}^n/\Gamma, \mathbb{Z})/S^U$ for a finite group S^U. For any $N \in \mathbb{N}$ there exists a normal subgroup $\Gamma_N \triangleleft \Gamma$ of finite index, such that the unramified covering $\varphi_N : \mathbb{B}^n/\Gamma_N \to \mathbb{B}^n/\Gamma$, induced by $\mathrm{Id}_{\mathbb{B}^n}$ extends to a covering $\varphi_N : (\mathbb{B}^n/\Gamma_N)' \to (\mathbb{B}^n/\Gamma)'$ with ramification index $> N$ over $(\mathbb{B}^n/\Gamma_N)' \setminus (\mathbb{B}^n/\Gamma_N)$. The argument exploits the residual finiteness of a lattice $\Gamma < SU_{n,1}$.

The torsion free $X' = (\mathbb{B}^2/\Gamma)'$ are shown to have geometric genus $p_g(X') = h^{2,0}(X') = 1$. The ones of Kodaira dimension $\kappa(X') \leq 1$ have irregularity $q(X') = h^{0,1}(X') \leq 2$ with $q(X') = 2$ exactly when X' is birational to an abelian surface. The torsion free $Y' = (\mathbb{B}^2/\Gamma_o)'$ of minimal $\mathrm{vol}(Y') = \mathrm{vol}(\mathbb{B}^2/\Gamma_o) = \frac{8\pi^2}{3}$ are characterized by the Kodaira-Enriques classification types of their minimal models Y, as well as by lower and upper bounds on number of the cusps of $\mathbb{B}^2/\Gamma_o$.

Let us recall some properties of the toroidal compactifications $X' = (\mathbb{B}^n/\Gamma)'$ of torsion free local complex hyperbolic spaces $\mathbb{B}^n/\Gamma$. Some references on the topic are Ash-Mumford-Rapoport-Tsai's [1], Mok's [19], Hummel's [17], Hummel-Schroeder's [16], Parker's [?] and McReynolds [22].

For an arbitrary boundary point

$$p = (p_1, \ldots, p_n) \in \partial \mathbb{B}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \quad | \quad |z_1|^2 + \ldots + |z_n|^2 < 1\},$$

the stabilizer

$$Stab(p) = \{g \in SU_{n,1} \quad g(p) = p\}$$

of p in $SU_{n,1}$ is a parabolic subgroup of $SU_{n,1}$. In order to describe the intersection $\Gamma(p) := \Gamma \cap Stab(p)$, let us denote by

$$\langle\langle\ ,\ \rangle\rangle : \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \longrightarrow \mathbb{C},$$

*Mathematics Subject Classification: Primary - 14M27; Secondary - 14J25.

Key words and phrases: Torsion free local complex hyperbolic spaces, toroidal compactification, covering of toroidal compactifications, fundamental group, integral homology groups, Kobayashi hyperbolic manifold.

Research partially supported by Contract 178 / 09. 05. 2012.

$$\langle\langle a,b\rangle\rangle=\sum_{i=1}^{n-1}a_i\overline{b_i}\quad \text{for}\quad \forall a,b\in\mathbb{C}^{n-1}$$

the standard Hermitian inner product on the complex vector space $\mathbb{C}^{n-1}$. The Heisenberg group $\mathcal{H}:=(\mathbb{C}^{n-1}\ltimes\mathbb{R},\circ)$ has composition law

$$(a,r)\circ(b,s):=(a+b,\quad r+s+2Im\langle\langle a,b\rangle\rangle)\quad \text{for}\quad \forall(a,r),(b,s)\in\mathbb{C}^{n-1}\times\mathbb{R}.$$

The neutral element of $\mathcal{H}$ is $(0^{n-1},0)$. The inverse of $(a,r)\in\mathcal{H}$ is $(a,r)^{-1}=(-a,-r)$. Consider the semi-direct product $\mathcal{A}:=(U_{n-1}\ltimes\mathcal{H},\star)$ with group law

$$(g,a,r)\star(h,b,s)=(gh,\ \ b+h^{-1}a,\ \ r+s+2Im\langle\langle a,hb\rangle\rangle)\quad \text{for}\quad \forall(g,a,r),(h,b,s)\in\mathcal{A}.$$

Its neutral element is $(I_{n-1},0^{n-1},0)$. Any $(g,a,r)\in\mathcal{A}$ has inverse $(g,a,r)^{-1}=(g^{-1},-ga,-r)\in\mathcal{A}$. The group $\mathcal{A}$ acts on $\mathcal{H}$ by the rule

$$(g,a,r)(b,s)=(g(a+b),\quad r+s+2Im\langle\langle a,b\rangle\rangle)\quad \text{for}\quad \forall(g,a,r)\in\mathcal{A},\ \ \forall(b,s)\in\mathcal{H}.$$

The commutant of $\mathcal{H}$ is $[\mathcal{H},\mathcal{H}]=(\{0^{n-1}\}\times\mathbb{R},\circ)\simeq(\mathbb{R},+)$. One can identify the abelianization map

$$Ab_{\mathcal{H}}:\mathcal{H}\longrightarrow\mathcal{H}/[\mathcal{H},\mathcal{H}]=(\mathbb{C}^{n-1}\times\{0\},\circ)\simeq(\mathbb{C}^{n-1},+),$$

$$Ab_{\mathcal{H}}(a,r)=(a,r)\circ[\mathcal{H},\mathcal{H}]=(a,0)\circ[\mathcal{H},\mathcal{H}]=a\quad \text{for}\quad \forall(a,r)\in\mathcal{H}$$

with the projection on $\mathbb{C}^{n-1}$.

The group $\Gamma(p):=\Gamma\cap Stab(p)$ is embedded in $\mathcal{A}$ (cf. McReynolds' [22]). The unipotent radical $W(p)$ of $Stab(p)$ is isomorphic to the Heisenberg group, so that the commutant $U(p)=[W(p),W(p)]$ of $W(p)$ can be identified with the subgroup $(\{I_{n-1}\}\ltimes(\{0^{n-1}\}\ltimes\mathbb{R}),\star)\simeq(\mathbb{R},+)$ of $\mathcal{A}$. If $p\in\partial_\Gamma\mathbb{B}^n$ is a Γ-rational boundary point then the intersection $\Gamma^U(p):=\Gamma\cap U(p)\simeq(\mathbb{Z},+)$ is a lattice of $(\mathbb{R},+)$. Let us denote by $r(p)$ the positive generator of $\Gamma\cap U(p)\simeq(r(p)\mathbb{Z},+)<(\mathbb{R},+)\simeq U(p)$. If $\Gamma^W(p):=\Gamma\cap W(p)$ then the quotient $\Lambda(p):=\Gamma^W(p)/\Gamma^U(p)\simeq\Gamma^W(p)U(p)/U(p)\simeq(\mathbb{Z}^{2n-2},+)$ is a lattice in $W(p)/U(p)\simeq(\mathbb{C}^{n-1},+)$.

In order to specify the $\Gamma(p)$-action on $\mathbb{B}^n$, let us introduce Siegel domain coordinates $t_1,\dots,t_{n-1},t_n$ at $p=(p_1,\dots,p_n)\in\partial\mathbb{B}^n$. Note that $\overline{p}=(\overline{p_1},\dots,\overline{p_n})\in\mathbb{C}^n$ is a unit vector with respect to the standard Hermitian inner product

$$\langle\ ,\ \rangle:\mathbb{C}^n\times\mathbb{C}^n\longrightarrow\mathbb{C},$$

$$\langle x,y\rangle=\sum_{i=1}^{n}x_i\overline{y_i}$$

on $\mathbb{C}^n$. Complete $\overline{p}\in\mathbb{C}^n$ to an orthonormal basis $q_1,\dots,q_{n-1},q_n:=\overline{p}\in\mathbb{C}^n$ of $\mathbb{C}^n$ and form the matrix

$$M=\begin{pmatrix}q_1\\ \dots\\ q_n\end{pmatrix}\in Mat_{n,n}(\mathbb{C}).$$

By its very definition, $M \in U_n$ is a unitary matrix. The $\mathbb{C}$-linear functionals

$$l_j : \mathbb{C}^n \longrightarrow \mathbb{C},$$

$$l_j(z) = \sum_{s=1}^{n} q_{j,s} z_s,$$

given by the rows $q_j = (q_{j1}, \dots, q_{jn})$ of M transform a point $z = (z_1, \dots, z_n) \in \mathbb{B}^n$ into a point $l(z) := (l_1(z), \dots, l_n(z)) \in \mathbb{B}^n$. Therefore

$$t_j := \frac{i l_j(z)}{(1 - l_n(z))} \quad \text{for} \quad 1 \leq j \leq n-1, \quad t_n := \frac{i(1 + l_n(z))}{(1 - l_n(z))}$$

are global holomorphic coordinates on the unit ball

$$\mathbb{B}^n = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n \ \Big| \ \sum_{j=1}^{n} |z_j|^2 < 1 \right\}$$

with respect to which

$$\mathbb{B}^n = \left\{ t = (t_1, \dots, t_{n-1}, t_n) \in \mathbb{C}^n \ \Big| \ Im(t_n) > \sum_{j=1}^{n-1} |t_j|^2 \right\}.$$

The action of $(g, a, r) \in \mathcal{A}$ on $\mathbb{B}^n$ in Siegel domain coordinates $(t', t_n) \in \mathbb{B}^n$ is

$$(g, a, r)(t', t_n) = (g(t' + a), \quad t_n + r + 2i\langle\langle t', a \rangle\rangle + i||a||^2).$$

The group $\Gamma^U(p) \simeq (r(p)\mathbb{Z}, +)$ acts on $\mathbb{C}^{n-1} \times \mathbb{C}$ by the rule

$$(r(p)z)(t', t_n) = (t', \quad t_n + r(p)z) \quad \text{for} \quad \forall z \in \mathbb{Z}.$$

The $\Gamma^U(p)$-Galois covering

$$\zeta_p^U : \mathbb{C}^{n-1} \times \mathbb{C} \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}^*,$$

$$\zeta_p^U(t', t_n) = \left(t', w = e^{\frac{2\pi i}{r(p)} t_n} \right)$$

transforms the ball $\mathbb{B}^n$ onto

$$\zeta_p^U(\mathbb{B}^n) = \mathbb{B}^n / \Gamma^U(p) = \left\{ (t', w) \in \mathbb{C}^{n-1} \times \mathbb{C} \ | \ 0 < |w|^2 < e^{-\frac{4\pi}{r(p)} ||t'||^2} \right\}.$$

For any $t' \in \mathbb{C}^{n-1}$, let us denote

$$\Delta^*(t') = \left\{ (t', w) \in \{t'\} \times \mathbb{C} \ | \ 0 < |w|^2 < e^{-\frac{4\pi}{r(p)} ||t'||^2} \right\}$$

and note that $\mathbb{B}^n / \Gamma^U(p) = \cup_{t' \in \mathbb{C}^{n-1}} \Delta^*(t')$ is a family of punctured discs $\Delta^*(t')$ of continuously variable radii over $\mathbb{C}^{n-1}$. The subgroup $(\{I_{n-1}\} \ltimes (\{0^{n-1}\} \ltimes \mathbb{R}), \star) \simeq (\mathbb{R}, +)$ of $\mathcal{A}$ is contained in the center of $\mathcal{A}$, so that $\Gamma^U(p)$ is a normal subgroup of

$\Gamma(p) < \mathcal{A}$. For any $(g, a, r) \in \Gamma(p)$ the coset $(g, a, r) \star \Gamma^U(p) = (g, a, 0) \star \Gamma^U(p)$ acts on $(t', w) \in \mathbb{B}^n/\Gamma^U(p)$ by the rule

$$[(g, a, 0) \star \Gamma^U(p)](t', w) = \left(g(t' + a), \quad we^{-\frac{4\pi}{r(p)}\langle\langle t', a\rangle\rangle - \frac{2\pi}{r(p)}||a||^2} \right). \tag{1}$$

This $\Gamma(p)/\Gamma^U(p)$-action on $\mathbb{B}^n/\Gamma^U(p)$ can be extended to $\mathbb{C}^{n-1} \times 0$ by

$$[(g, a, 0) \star \Gamma^U(p)](t', 0) = \left(g(t' + a), \quad 0 \right)$$

for all $(g, a, 0) \star \Gamma^U(p) \in \Gamma(p)/\Gamma^U(p)$ and all $\forall (t', 0) \in \mathbb{C}^{n-1} \times 0$. Note that the union $\mathbb{B}^n/\Gamma^U(p) \cup (\mathbb{C}^{n-1} \times 0) = \cup_{t' \in \mathbb{C}^{n-1}} \Delta(t')$ is a family of discs

$$\Delta(t') = \left\{ (t', w) \in \{t'\} \times \mathbb{C} \ \mid \ |w|^2 < e^{-\frac{4\pi}{r(p)}||t'||^2} \right\}$$

of continuously variable radii over $\mathbb{C}^{n-1}$. Any coset $(a, 0) \circ \Gamma^U(p) = (I_{n-1}, a, 0) \star \Gamma^U(p) \in \Lambda(p) = \Gamma^W(p)/\Gamma^U(p) < \Gamma(p)/\Gamma^U(p)$ transforms $\Delta^*(t')$ onto $\Delta^*(t' + a)$ and $\Delta(t')$ onto $\Delta(t' + a)$. Therefore $\mathbb{B}^n/\Gamma^W(p)$ is a family of punctured discs over the compact complex torus $T(p) := (\mathbb{C}^{n-1}, +)/(\Lambda(p), +)$. Filling in the centers of these punctured discs, one obtains the family $\mathbb{B}^n/\Gamma^W(p) \cup T(p)$ of discs over $T(p)$.

For an arbitrary $N \in \mathbb{N}$, let us consider the horoball neighborhood

$$\mathbb{B}^n(p, N) := \{(t', t_n) \in \mathbb{C}^n \ \mid \ Im(t_n) > ||t'||^2 + N\} =$$

$$= \left\{ z \in \mathbb{C}^n \ \middle| \ \frac{|1 - l_n(z)|^2}{1 - \sum\limits_{j=1}^{n} |l_j(z)|^2} < \frac{1}{N} \right\}$$

of $p \in \partial_\Gamma \mathbb{B}^n$ on $\mathbb{B}^n$. The group $\mathcal{A}$ acts on $\mathbb{B}^n(p, N)$. Therefore $\Gamma(p)$ acts on $\mathbb{B}^n(p, N)$. The quotient

$$Z(p, N) := \mathbb{B}^n(p, N)/\Gamma^W(p) = \cup_{t' + \Lambda(p) \in T(p)} \Delta^*_N(t' + \Lambda(p))$$

is a family of punctured discs over the torus $T(p)$ and

$$\widehat{Z(p, N)} := Z(p, N) \cup T(p) = \cup_{t' + \Lambda(p) \in T(p)} \Delta_N(t' + \Lambda(p))$$

is a family of discs over $T(p)$. It is well known that for a sufficiently large $N \in \mathbb{N}$, the horoballs $\mathbb{B}^n(\gamma(p), N)$ with $\gamma \in \Gamma \setminus Stab(p)$ are pairwise disjoint.

Lemma 1. *Let Γ be a torsion free lattice of $SU_{n,1}$, $p \in \partial_\Gamma \mathbb{B}^n$ be a Γ-rational boundary point over the cusp $\kappa \in \partial_\Gamma \mathbb{B}^n/\Gamma$, $\Gamma^W(p) := \Gamma \cap W(p)$ be the intersection of Γ with the unipotent radical $W(p)$ of the stabilizer $Stab(p)$ of p in $SU_{n,1}$ and $\eta_p : \mathbb{B}^n/\Gamma^W(p) \to \mathbb{B}^n/\Gamma$ be the covering, induced by the identity of $\mathbb{B}^n$. Then for any $N \in \mathbb{N}$ with $\mathbb{B}^n(\gamma_1(p), N) \cap \mathbb{B}^n(\gamma_2(p), N) = \emptyset$ for all $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1^{-1}\gamma_2 \notin Stab(p)$, the restriction*

$$\eta_p : Z(p, N) = \mathbb{B}^n(p, N)/\Gamma^W(p) \longrightarrow \eta_p(Z(p, N)) =: V(\kappa, N) \subset \mathbb{B}^n/\Gamma$$

is an isomorphism onto its image $V(\kappa, N)$ and extends to an isomorphism

$$\eta_p : \widehat{Z(p, N)} = Z(p, N) \cup T(p) \longrightarrow \widehat{V(\kappa, N)} = V(\kappa, N) \cup \eta_p(T(p)).$$

Proof. Let $\Gamma(p) := \Gamma \cap Stab(p)$ be the intersection of Γ with the stabilizer of p. The normal subgroup $W(p) \simeq \mathcal{H}$ of $\mathcal{A}$ intersects the lattice Γ in a normal subgroup $\Gamma^W(p)$ of $\Gamma(p)$ and the quotient $\Gamma(p)/\Gamma^W(p)$ acts on $Z(p,N)$. According to (1), any $(g,a,r)\star\Gamma^W(p) = (g,0^{n-1},0)\star\Gamma^W(p) \in \Gamma(p)/\Gamma^W(p)$ acts on $(t'+\Lambda(p),w) \in Z(p,N)$ by the rule

$$[(g,0^{n-1},0)\star\Gamma^W(p)](t'+\Lambda(p),w) = (gt'+g\Lambda(p),w).$$

This action extends to $(t'+\Lambda(p),0) \in T(p)$ by

$$[(g,0^{n-1},0)\star\Gamma^W(p)](t'+\Lambda(p),0) = (gt'+g\Lambda(p),0).$$

Note that $(g\Lambda(p),+) < (\mathbb{C}^{n-1},+)$ is a free $\mathbb{Z}$-module of rank $2n-2$ and the quotient $T^g(p) := \mathbb{C}^{n-1}/g\Lambda(p)$ is a compact complex torus. The map

$$g : T(p) = \mathbb{C}^{n-1}/\Lambda(p) \longrightarrow T^g(p) = \mathbb{C}^{n-1}/g\Lambda(p),$$

$$g(t'+\Lambda(p)) = gt'+g\Lambda(p)$$

is an isomorphism. The $\Gamma(p)/\Gamma^W(p)$-orbit of $T(p)$ is a disjoint union of compact complex tori $T^g(p)$, $g \in U_{n-1}$. In general, the tori $T^g(p)$ are not attached at Γ-rational boundary points of $\mathbb{B}^n$. The $\Gamma(p)/\Gamma^W(p)$-Galois covering

$$\zeta_p^W : Z(p,N) := \mathbb{B}^n(p,N)/\Gamma^W(p) \longrightarrow Y(p,N) := \mathbb{B}^n(p,N)/\Gamma(p)$$

extends to an isomorphism

$$\zeta_p^W : \widehat{Z(p,N)} = Z(p,N) \cup T(p) \longrightarrow \widehat{Y(p,N)} = Y(p,N) \cup \zeta_p^W(T(p))$$

exactly when for $\forall (g,0^{n-1},0)\star\Gamma^W(p) \in \Gamma(p)/\Gamma^W(p) \setminus \{\Gamma^W(p)\}$ and $\forall (c+\Lambda(p),w) \in \widehat{Z(p,N)}$ the point $[(g,0^{n-1},0)\star\Gamma^W(p)](c+\Lambda(p),w) = (gc+g\Lambda(p),w)$ does not belong to $\widehat{Z(p,N)}$. Assume that there exist $[(g,0^{n-1},0)\star\Gamma^W(p)] \in \Gamma(p)/\Gamma^W(p) \setminus \{\Gamma^W(p)\}$ and $c+\Lambda(p) \in T(p)$ with $gc+g\Lambda(p) \in T(p) = \mathbb{C}^{n-1}/\Lambda(p)$. Then $g\Lambda(p) = \Lambda(p)$, as far as the tori with different lattices (fundamental groups) are disjoint in the universal family of the compact complex $(n-1)$-dimensional tori. Then for $\forall (\Lambda(p),w) \in Z(p,N) = \mathbb{B}^n(p,N)/\Gamma^W(p)$ one has $[(g,0^{n-1},0)\star\Gamma^W(p)](\Lambda(p),w) = (\Lambda(p),w)$, so that all the points $(\Lambda(p),w) \in Z(p,N)$ from the punctured disc over the origin $\Lambda(p)$ of $T(p)$ are fixed by $\Gamma(p)/\Gamma^W(p)$. As a result, the covering $\zeta_p^W : \mathbb{B}^n(p,N)/\Gamma^W(p) \to \mathbb{B}^n(p,N)/\Gamma(p)$ is ramified. However, for a torsion free lattice $\Gamma < SU_{n,1}$, the cusp group $\Gamma(p) = \Gamma \cap Stab(p)$ has no fixed points on $\mathbb{B}^n(p,N)$ and the covering $\mathbb{B}^n(p,N) \to \mathbb{B}^n(p,N)/\Gamma(p)$ is unramified. According to the factorization

$$\begin{tikzcd}
\mathbb{B}^n(p,N) \arrow[d] \arrow[dr] & \\
\mathbb{B}^n(p,N)/\Gamma^W(p) \arrow[r] & \mathbb{B}^n(p,N)/\Gamma(p)
\end{tikzcd},$$

the covering $\mathbb{B}^n(p,N)/\Gamma^W(p) \to \mathbb{B}^n/\Gamma(p)$ is to be unramified. The contradiction justifies that $g\Lambda(p) \neq \Lambda(p)$ for $\forall (g, 0^{n-1}, 0) \star \Gamma^W(p) \in \Gamma(p)/\Gamma^W(p) \setminus \{\Gamma^W(p)\}$. Thus, for any $(g, 0^{n-1}, 0) \star \Gamma^W(p) \in \Gamma(p)/\Gamma^W(p) \setminus \{\Gamma^W(p)\}$ and any $(c+\Lambda(p), w) \in \widehat{Z(p,N)}$ one has $[(g, 0^{n-1}, 0) \star \Gamma^W(p)](c+\Lambda(p), w) = (gc + g\Lambda(p), w) \notin \widehat{Z(p,N)}$ and the map $\zeta_p^W : \widehat{Z(p,N)} \to \widehat{Y(p,N)}$ is an isomorphism.

Further, consider the unramified (not necessarily Galois) covering

$$\zeta_p : \mathbb{B}^n/\Gamma(p) \longrightarrow \mathbb{B}^n/\Gamma$$

and the cusp $\kappa \in \partial_\Gamma \mathbb{B}^n/\Gamma$, representing the Γ-orbit of $p \in \partial_\Gamma \mathbb{B}^n$. For sufficiently large $N \in \mathbb{N}$, the horoballs $\mathbb{B}(q,N)$, centered at $q \in \partial_\Gamma \mathbb{B}^n$ are disjoint and

$$\zeta_p : Y(p,N) = \mathbb{B}^n(p,N)/\Gamma(p) \longrightarrow \mathbb{B}^n/\Gamma$$

is biholomorphic onto its image $V(\kappa,N) := \zeta_p\left(\mathbb{B}^n(p,N)/\Gamma(p)\right)$. The map ζ_p is locally bounded around $T(p)$ and extends to a biholomorphism

$$\zeta_p : \widehat{Y(p,N)} \equiv Z(p,N) \cup T(p) \longrightarrow \widehat{V(\kappa,N)} := V(\kappa,N) \cup T(\kappa)$$

with $T(\kappa) := \zeta_p \circ \zeta_p^W(T(p)) \simeq T(p)$. As a result, the composition

$$\eta_p := \zeta_p \circ \zeta_p^W : \widehat{Z(p,N)} \longrightarrow \widehat{V(\kappa,N)}$$

is an isomorphism, which restricts to isomorphisms $\eta_p : Z(p,N) \to V(\kappa,N)$ and $\eta_p : T(p) \to T(\kappa)$.

$\square$

The neighborhoods $\widehat{V(\kappa,N)}$ of the irreducible components $T(\kappa)$ of the toroidal compactifying divisor $T = (\mathbb{B}^n/\Gamma)' \setminus (\mathbb{B}^n/\Gamma)$ of $\mathbb{B}^n/\Gamma$ are disjoint for sufficiently large $N \in \mathbb{N}$ and

$$X' = (\mathbb{B}^n/\Gamma) \cup \left(\coprod_{\kappa \in \partial_\Gamma \mathbb{B}^n/\Gamma} \widehat{V(\kappa,N)} \right) \tag{2}$$

is an open cover of the toroidal compactification X'.

From now on, for an arbitrary lattice G of $SU_{n,1}$ we denote by ζ_G the G-Galois coverings $\zeta_G : \mathbb{B}^n \to \mathbb{B}^n/G$ and $\zeta_G : \partial_G \mathbb{B}^n \to \partial_G \mathbb{B}^n/G$.

1 The fundamental group of the toroidal compactification of a torsion free local complex hyperbolic space

Theorem 2. *Let $X' = (\mathbb{B}^n/\Gamma)'$ be the toroidal compactification of a torsion free local complex hyperbolic space $\mathbb{B}^n/\Gamma$. For any Γ-rational boundary point $p \in \partial_\Gamma \mathbb{B}^n$, denote by $U(p) = [W(p), W(p)]$ the commutant of the unipotent radical $W(p)$ of $Stab(p) = \{g \in SU_{n,1} \mid g(p) = p\}$ and put $\Gamma^U(p) := \Gamma \cap U(p) \simeq (\mathbb{Z}, +)$. Then the fundamental group*

$$\pi_1(X') \simeq \Gamma/\Gamma^U \simeq \Gamma/\langle \Gamma^U(p) \mid p \in \partial_\Gamma \mathbb{B}^n \rangle$$

is isomorphic to the quotient of the lattice Γ *by the normal subgroup* Γ^U*, generated by* $\Gamma^U(p)$ *for all the* Γ*-rational boundary points* $p \in \partial_\Gamma \mathbb{B}^n$.

Proof. Let $\kappa = \zeta_\Gamma(p) \in \partial_\Gamma \mathbb{B}^n/\Gamma$ be a Γ-cusp and $p \in \partial_\Gamma \mathbb{B}^n$ be a Γ-rational boundary point over κ. The biholomorphism $\eta_p := \zeta_p \circ \zeta_p^W : Z(p,N) \to V(\kappa,N)$ allows to identify the fundamental groups $\pi_1(V(\kappa,N)) = \pi_1(Z(p,N))$. By the simply connectedness of the horoball $\mathbb{B}^n(p,N)$ and the lack of $\Gamma^W(p)$-fixed points on $\mathbb{B}^n(p,N)$, one has $\pi_1(Z(p,N)) = \pi_1(\mathbb{B}^n(p,N)/\Gamma^W(p)) = \Gamma^W(p)$. The presence of a biholomorphism $\eta_p : \widehat{Z(p,N)} \to \widehat{V(\kappa,N)}$ allows to assume that $\pi_1(\widehat{V(\kappa,N)}) = \pi_1(\widehat{Z(p,N)})$. The fibration $\widehat{Z(p,N)}$ by discs of continuously variable radii over $T(p)$ is homotopy equivalent to $T(p)$ and $\pi_1(\widehat{Z(p,N)} = \pi_1(T(p)) = \Lambda(p) = \Gamma^W(p)/\Gamma^U(p)$. The identical inclusion $V(\kappa,N) \hookrightarrow \widehat{V(\kappa,N)}$ induces the epimorphism $\Gamma^W(p) \to \Gamma^W(p)/\Gamma^U(p)$, with cyclic kernel $\Gamma^U(p) = \langle c(p) \rangle$.

Let $\partial_\Gamma \mathbb{B}^n/\Gamma = \{\kappa_i \ | \ 1 \leq i \leq h\}$ be the cusps of $\mathbb{B}^n/\Gamma$, $T_i := T(\kappa_i)$, $1 \leq i \leq h$ be the corresponding irreducible components of the toroidal compactifying divisor $T = (\mathbb{B}^n/\Gamma)' \setminus (\mathbb{B}^n/\Gamma)$. Fix Γ-rational boundary points $p_i \in \zeta_\Gamma^{-1}(\kappa_i)$ over the cusps κ_i. Put $X_0 := \mathbb{B}^n/\Gamma$, $X_i := (\mathbb{B}^n/\Gamma) \cup \left(\coprod_{j=1}^{i} \widehat{V(\kappa_j,N)} \right)$ for $1 \leq i \leq h$ and note that

$$X_0 \subset X_1 \subset \ldots \subset X_{h-1} \subset X_h = X'$$

is an increasing sequence of open subsets of X'.

By an induction on $1 \leq i \leq h$, we claim that the fundamental group $\pi_1(X_i) \simeq \Gamma/\langle c(p_j) \ | \ 1 \leq j \leq i \rangle\rangle$ is isomorphic to the quotient group of Γ by the normal subgroup, generated by $c(p_1), \ldots, c(p_i)$. To this end, note that if G is a group, F is a normal subgroup of G, H is a subgroup of G and K is a normal subgroup of H, then the amalgamated product

$$(G/F) *_H (H/K) \simeq G/(F\langle\langle K \rangle\rangle),$$

is the quotient of G by the product of F and the normal subgroup $\langle\langle K \rangle\rangle$ of G, generated by K. More precisely, for any presentation $H = \langle Gen(H) \ \ | Rel(H) \rangle$ of H with generators $Gen(H)$ and relations $Rel(H)$, there is a presentation $H/K = \langle Gen(H) \ \ | Rel(H), K \rangle$ of H/K. The amalgamated product $\mathcal{P} := (G/F) *_H (H/K)$ is generated by $G \cup Gen(H)$. Any generator h of H/K is identified with the coset $hF \in G/F$. Thus, G generates $\mathcal{P}$. The relations $Rel(H)$ of the subgroup H of G follow automatically from the relations of G and hold in G/F. One has to impose only the relations from K on the subgroup of G/F, generated by hF for $h \in Gen(H)$. In other words, $\mathcal{P} = G/(F\langle\langle K \rangle\rangle)$.

In the case under consideration, $X_1 = X_0 \cup \widehat{V(\kappa_1,N)}$ is the union of the path connected open subsets $X_0 = \mathbb{B}^n/\Gamma$, $\widehat{V(\kappa_1,N)}$ with $(\mathbb{B}^n/\Gamma) \cap \widehat{V(\kappa_1,N)} = V(\kappa_1,N)$. By Seifert-van Kampen Theorem,

$$\pi_1(X_1) = \Gamma *_{\pi_1(V(\kappa_1,N))} \pi_1(\widehat{V(\kappa_1,N)}).$$

Bearing in mind that $\pi_1(\widehat{V(\kappa_1, N)}) = \pi_1(V(\kappa_1, N))/\langle\langle c(p_1)\rangle\rangle$, one applies the previous considerations and concludes that $\pi_1(X_1) = \Gamma/\langle\langle c(p_1)\rangle\rangle$. In general, suppose that

$$\pi_1(X_{i-1}) = \Gamma/\langle\langle c(p_j) \mid 1 \leq j \leq i-1\rangle\rangle.$$

Then $X_i = X_{i-1} \cup \widehat{V(\kappa_i, N)}$ is the union of the path connected open subsets X_{i-1}, $\widehat{V(\kappa_i, N)}$ with $X_{i-1} \cap \widehat{V(\kappa_i, N)} = V(\kappa_i, N)$. By Seifert-van Kampen Theorem and the above considerations,

$$\pi_1(X_i) = \pi_1(X_{i-1}) *_{\pi_1(V(\kappa_i,N))} \pi_1(\widehat{V(\kappa_i, N)}) =$$

$$= \Gamma/\langle\langle c(p_j) \mid 1 \leq j \leq i-1\rangle\rangle *_{\pi_1(V(\kappa_i,N))} \pi_1(V(\kappa_i, N))/\langle\langle c(p_i)\rangle\rangle =$$

$$= \Gamma/(\langle\langle c(tbp_j) \mid 1 \leq j \leq i-1\rangle\rangle\langle\langle c(p_i)\rangle\rangle) = \Gamma/\langle\langle c(p_j) \mid 1 \leq j \leq i\rangle\rangle.$$

In particular, $\pi_1(X') = \pi_1(X_h) = \Gamma/\langle\langle c(p_j) \mid 1 \leq j \leq h\rangle\rangle$.

The normal subgroup $\langle\langle c(p_j) \mid 1 \leq j \leq h\rangle\rangle$ of Γ, generated by $c(p_j)$ coincides with the subgroup of Γ, generated by $c(p)$ for all $p \in \partial_\Gamma \mathbb{B}^n$. Indeed, for any $\gamma \in \Gamma$ there holds $\gamma Stab(p)\gamma^{-1} = Stab(\gamma p_j)$. Therefore $\gamma W(p_j)\gamma^{-1} = W(\gamma p_j)$, as far as the unipotency of an element is invariant under conjugation. Further,

$$\gamma U(p_j)\gamma^{-1} = \gamma[W(p_j), W(p_j)]\Gamma^{-1} =$$

$$= [\gamma W(p_j)\gamma^{-1}, \gamma W(p_j)\gamma^{-1}] = [W(\gamma p_j), W(\gamma p_j)] = U(\gamma p_j)$$

implies that

$$\Gamma^U(\gamma p_j) = \Gamma \cap U(\gamma p_j) = \Gamma \cap \gamma U(p_j)\gamma^{-1} =$$

$$= \gamma(\Gamma \cap U(p_j))\gamma^{-1} = \gamma\Gamma^U(p_j)\gamma^{-1} = \langle \gamma c(p_j)\gamma^{-1}\rangle,$$

so that one can choose generators $c(\gamma p_j) := \gamma c(p_j)\gamma^{-1}$ of $\Gamma^U(\gamma p_j)$ for all $\gamma \in \Gamma$. Bearing in mind that the disjoint union $\coprod_{j=1}^{h} Orb_\Gamma(p_j) = \partial_\Gamma \mathbb{B}^n$ of the Γ-orbits of $p_1, \ldots, p_h$ coincides with the set $\partial_\Gamma \mathbb{B}^n$ of the Γ-rational boundary points, one concludes that the normal subgroup of Γ, generated by $c(p_j)$ with $1 \leq j \leq h$ coincides with the subgroup $\langle c(p) \mid p \in \partial_\Gamma \mathbb{B}^n\rangle = \langle \Gamma^U(p) \mid p \in \partial_\Gamma \mathbb{B}^n\rangle$.

$\square$

Corollary 3. *Let Γ be a torsion free lattice of $SU_{n,1}$, $[\Gamma, \Gamma]$ be the commutant of Γ and Γ^U be the normal subgroup of Γ, generated by the intersections $\Gamma^U(p) := \Gamma \cap U(p)$ of Γ with the commutants $U(p) = [W(p), W(p)]$ of the unipotent radicals $W(p)$ of the stabilizers $Stab(p) < SU_{n,1}$ of all the Γ-rational boundary points $p \in \partial_\Gamma \mathbb{B}^n$. Then the firstintegral homology group*

$$H_1((\mathbb{B}^n/\Gamma)', \mathbb{Z}) \simeq H_1(\mathbb{B}^n/\Gamma, \mathbb{Z})/S^U$$

of the toroidal compactification $(\mathbb{B}^n/\Gamma)'$ of $\mathbb{B}^n/\Gamma$ is the quotient of the first integral homology group of $\mathbb{B}^n/\Gamma$ by the finite subgroup $S^U = (\Gamma^U[\Gamma, \Gamma])/[\Gamma, \Gamma]$.

In particular, $\mathrm{rk} H_1((\mathbb{B}^n/\Gamma)', \mathbb{Z}) = \mathrm{rk} H_1(\mathbb{B}^n/\Gamma, \mathbb{Z})$.

Proof. Let M be a manifold with fundamental group $\pi_1(M)$. Then the first homology group of M with integral coefficients

$$H_1(M, \mathbb{Z}) \simeq ab(\pi_1(M)) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$$

is isomorphic to the abelianization $ab(\pi_1(M))$ of $\pi_1(M)$, i.e., to the quotient of $\pi_1(M)$ by its commutant $[\pi_1(M), \pi_1(M)]$. According to Theorem 2, the toroidal compactification $X' = (\mathbb{B}^n/\Gamma)'$ of $\mathbb{B}^n/\Gamma$ has fundamental group $\pi_1(X') \simeq \Gamma/\Gamma^U$. The commutant $[\Gamma/\Gamma^U, \Gamma/\Gamma^U]$ of Γ/Γ^U is generated by $[\gamma_1, \gamma_2]\Gamma^U = (\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})\Gamma^U$ for $\forall \gamma_1, \gamma_2 \in \Gamma$ and coincides with $([\Gamma, \Gamma]\Gamma^U)/\Gamma^U$. Therefore the first integral homology group

$$\begin{aligned} H_1(X', \mathbb{Z}) = ab(\pi_1(X')) \simeq (\Gamma/\Gamma^U)/[\Gamma/\Gamma^U, \Gamma/\Gamma^U] \simeq \\ \simeq (\Gamma/\Gamma^U)/([\Gamma, \Gamma]\Gamma^U/\Gamma^U) \simeq \Gamma/([\Gamma, \Gamma]\Gamma^U). \end{aligned} \tag{3}$$

Note that $[\Gamma, \Gamma]$ and Γ^U are normal subgroups of Γ, so that their product $[\Gamma, \Gamma]\Gamma^U = \Gamma^U[\Gamma, \Gamma] = \{\alpha\beta \mid \alpha \in [\Gamma, \Gamma], \ \beta \in \Gamma^U\}$ is a normal subgroup of Γ, containing $[\Gamma, \Gamma]$. On the other hand, $\pi_1(\mathbb{B}^n/\Gamma) = \Gamma$, as far as the torsion free lattice $\Gamma < SU_{n,1}$ has no fixed points on $\mathbb{B}^n$. Therefore

$$H_1(\mathbb{B}^n/\Gamma, \mathbb{Z}) = ab(\Gamma) = \Gamma/[\Gamma, \Gamma]. \tag{4}$$

Combining (3) with (4), one concludes that

$$H_1(X', \mathbb{Z}) \simeq \Gamma/([\Gamma, \Gamma]\Gamma^U) \simeq (\Gamma/[\Gamma, \Gamma])/([\Gamma, \Gamma]\Gamma^U/[\Gamma, \Gamma]) \simeq H_1(\mathbb{B}^n/\Gamma, \mathbb{Z})/S^U$$

for the subgroup $S^U := ([\Gamma, \Gamma]\Gamma^U)/[\Gamma, \Gamma]$ of $H_1(\mathbb{B}^n/\Gamma, \mathbb{Z})$. There remains to be shown that S^U is finite. To this end, let $\kappa_1, \ldots, \kappa_h \in \partial_\Gamma \mathbb{B}^n/\Gamma$ be the Γ-cusps and $p_i \in \zeta_\Gamma^{-1}(\kappa_i) \subseteq \partial_\Gamma \mathbb{B}^n$ be Γ-rational boundary points over κ_i for all $1 \leq i \leq h$. If $c(p_i)$ generate $\Gamma^U(p_i) := \Gamma \cap U(p_i)$ then

$$\Gamma^U = \langle \gamma c(p_i)\gamma^{-1} \mid \gamma \in \Gamma, \ \ 1 \leq i \leq h \rangle$$

and $S^U = \Gamma^U[\Gamma, \Gamma]/[\Gamma, \Gamma]$ is generated by $\{\gamma c(p_i)\gamma^{-1}[\Gamma, \Gamma] \mid \gamma \in \Gamma, \ \ 1 \leq i \leq h\}$. Bearing in mind that

$$\gamma c(p_i)\gamma^{-1}[\Gamma, \Gamma] = c(p_i)(c(p_i)^{-1}\gamma c(p_i)\gamma^{-1})[\Gamma, \Gamma] = c(p_i)[c(p_i)^{-1}, \gamma][\Gamma, \Gamma] = c(p_i)[\Gamma, \Gamma],$$

one concludes that

$$S^U = \langle c(p_i)[\Gamma, \Gamma] \mid 1 \leq i \leq h \rangle$$

is a finitely generated abelian group. It suffices to show that any $c(p_i)[\Gamma, \Gamma] \in S^U$ is of finite order, in order to conclude that S^U is a finite group. To this end, let us recall that the unipotent radical $W(p_i)$ of $Stab(p_i)$ is isomorphic to the Heisenberg group $\mathcal{H} = (\mathbb{C}^{n-1} \ltimes \mathbb{R}, \circ)$ and its commutant

$$U(p_i) = [W(p_i), W(p_i)] \simeq (\{0^{n-1} \ltimes \mathbb{R}, \circ) \simeq (\mathbb{R}, +).$$

Denote by $r(p_i) \in \mathbb{R}^*$ the non-zero real number, corresponding to the generator $c(p_i)$ of $\Gamma^U(p_i) = \Gamma \cap U(p_i) \simeq (r(p_i)\mathbb{Z}, +) < (\mathbb{R}, +)$. If $\Gamma^W(p_i) := \Gamma \cap W(p_i)$, then the

quotient $\Lambda(p_i) = \Gamma^W(p_i)/\Gamma^U(p_i) \simeq (\mathbb{Z}^{2n-2},+)$ is a lattice in $\mathbb{C}^{n-1}$. For an arbitrary $\lambda \in \Lambda(p_i) \setminus \{0^{n-1}\} \subset \mathbb{C}^{n-1}$ we claim the existence of $\mu \in \Lambda(p_i) \subset \mathbb{C}^{n-1}$ with $Im\langle\langle \lambda, \mu\rangle\rangle \neq 0$. Otherwise, the set

$$\Sigma(\lambda) := \{x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1} \ \mid \ Im\langle\langle \lambda, x\rangle\rangle = 0\} =$$

$$= \{x \in \mathbb{C}^{n-1} \ \mid \ \langle\langle \lambda, x\rangle\rangle = \langle\langle x, \lambda\rangle\rangle\}$$

contains the lattice $\Lambda(p_i)$. One can view $\Sigma(\lambda)$ with $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$ as a real hyperplane

$$\Sigma(\lambda) = \Big\{(Re(x_1), Im(x_1), \ldots, Re(x_{n-1}), Im(x_{n-1})) \in \mathbb{R}^{2n-2} \ \mid$$

$$= \sum_{s=1}^{n-1} Im(\lambda_s)Re(x_s) - \sum_{s=1}^{n-1} Re(\lambda_s)Im(x_s) = 0\Big\}$$

in $\mathbb{R}^{2n-2}$ through the origin. Therefore $\Sigma(\lambda)$ is a real vector space of dimension $2n-3$. If the lattice $\Lambda(p_i)$ of $\mathbb{C}^{n-1}$ is contained in $\Sigma(\lambda)$, then its real span $\mathrm{Span}_{\mathbb{R}}(\Lambda(p_i)) = \mathbb{C}^{n-1} \simeq \mathbb{R}^{2n-2}$ is contained in $\Sigma(\lambda) \simeq \mathbb{R}^{2n-3}$. The contradiction justifies that for $\forall \lambda \in \Lambda(p_i) \setminus \{0^{n-2}\}$ there exists $\mu \in \Lambda(p_i)$ with $Im\langle\langle \lambda, \mu\rangle\rangle \neq 0$. For arbitrary liftings $(\lambda, r_i'), (\mu, r_i'') \in \Gamma^W(p_i)$ of $\lambda, \mu \in \Lambda(p_i) = \Gamma^W(p_i)/\Gamma^U(p_i)$, note that the commutator

$$[(\lambda, r_i'), (\mu, r_i'')] = (\lambda, r_i') \circ (\mu, r_i'') \circ (\lambda, r_i')^{-1} \circ (\mu, r_i'')^{-1} =$$

$$= (0^{n-1}, 4Im\langle\langle \lambda, \mu\rangle\rangle) \in [\Gamma^W(p_i), \Gamma^W(p_i)] \subseteq \Gamma^U(p_i) \cap [\Gamma, \Gamma].$$

Therefore $4Im\langle\langle \lambda, \mu\rangle\rangle = z_i r(p_i)$ for some $0 \neq z_i \in \mathbb{Z}$ and $c(p_i)^{|z_i|} \in [\Gamma, \Gamma]$ with $|z_i| \in \mathbb{N}$. Thus, $c(p_i)[\Gamma, \Gamma] \in S^U$ are of finite order and S^U is a finite group.

□

2 Geometric properties of the toroidal compactifications of torsion free local complex hyperbolic surfaces

The present section elaborates on the impact of the toroidal compactifying divisor $T = (\mathbb{B}^2/\Gamma)' \setminus (\mathbb{B}^2/\Gamma)$ of a torsion free local complex hyperbolic surface $\mathbb{B}^2/\Gamma$ on the geometric properties of $X' = (\mathbb{B}^2/\Gamma)'$.

Lemma 4. *Let $X' = (\mathbb{B}^2/\Gamma)'$ be the toroidal compactification of a torsion free $\mathbb{B}^2/\Gamma$. Then:*

(i) X' is a projective surface without Kobayashi hyperbolic smooth models;

(ii) if a smooth model of X' fibers over a smooth irreducible curve C then C is either elliptic or rational;

(iii) there is a blow-up S of X' at finitely many points, which fibers over a smooth irreducible elliptic or rational curve.

Proof. (i) If the compact complex surface X' is not projective, then the minimal model X of X' is of Class VII, primary or secondary Kodaira surface. In either case, the Kodaira dimension $\kappa(X') \leq 0$ and the irregularity $q(X') = h^{0,1}(X) \geq 1$. Then by Momot's [20], X is to be an abelian surface. The contradiction shows that X and any birational model of X is a projective surface.

Let us assume that there is a Kobayashi hyperbolic birational model Z of X'. Then there exists a surface Y and blow-ups

$$X' \xleftarrow{\sigma_1} Y \xrightarrow{\sigma_2} Z$$

at finitely many points. The proper transform F of $T = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ under σ_1 consists of smooth elliptic irreducible components F_i. The images $D_i = \sigma_2(F_i)$ of F_i under σ_2 are smooth elliptic curves, as far as σ_2 restricts to morphisms of the smooth curves F_i. The universal coverings $U_i : \widetilde{D_i} = \mathbb{C} \to D_i$ of $D_i \subset Z$ are non-constant holomorphic maps $U_i : \mathbb{C} \to Z$. By Brody's Theorem (cf.[5]), that contradicts Kobayashi hyperbolicity of Z.

(ii) Let us suppose that a smooth model Z of X' admits a fibration $f : Z \to C$ over a smooth irreducible curve C. Then there are blow-ups

$$X' \xleftarrow{\sigma_1} Y \xrightarrow{\sigma_2} Z$$

at finitely many points. If $E(\sigma_j)$ is the exceptional divisor of σ_j then $X \setminus \sigma_1 E(\sigma_1) \equiv Y \setminus E(\sigma_1)$ and $Y \setminus E(\sigma_2) \equiv Z \setminus \sigma_2 E(\sigma_2)$. Therefore

$$X \setminus (\sigma_1 E(\sigma_1) \cup \sigma_1 E(\sigma_2) \equiv Y \setminus (E(\sigma_1) \cup E(\sigma_2)) \equiv Z \setminus (\sigma_2 E(\sigma_1) \cup \sigma_2 E(\sigma_2)),$$

where $\sigma_j E(\sigma_j)$ are finite sets of points and $\sigma_j E(\sigma_{3-j})$, $1 \leq j \leq 2$ are divisors with rational irreducible components. As in the proof of (i), denote by F the proper transform of $T = X' \setminus (\mathbb{B}^2/\Gamma) = \sigma_1(F)$ under σ_1 and put $D = \sigma_2(F)$ for the image of F under σ_2. Then

$$(\mathbb{B}^2/\Gamma) \setminus (\sigma_1 E(\sigma_1) \cup \sigma_1 E(\sigma_2)) \equiv Y \setminus (E(\sigma_1) \cup E(\sigma_2) \cup F) \equiv Z \setminus (\sigma_2 E(\sigma_1) \cup \sigma_2 E(\sigma_2) \cup D)$$

is an open Kobayashi hyperbolic surface.

Assume that the fibration $f : Z \to C$ has base C of genus $g(C) \geq 2$. If the generic fibre of f is of genus at least 2 then by Shabat's thesis [23], the universal covering $\widetilde{Z}$ os Z is a bounded domain in $\mathbb{C}^2$. As a consequence, Z is to be Kobayashi hyperbolic. That contradicts (i) and reduces the considerations to a ruled surface $f : Z \to C$ or an elliptic fibration $f : Z \to C$. In either case, the generic fibre of f is not Kobayashi hyperbolic.The morphism f shrinks to points the smooth elliptic irreducible components D_i, $1 \leq i \leq h$ of D and the rational components of $\sigma_2 E(\sigma_1)$, as far as the genus of C is $g(C) \geq 2$. Therefore $\sigma_2 E(\sigma_1) \cup \sigma_2 E(\sigma_2) \cup D$ is contained in finitely many fibres of $f : Z \to C$ and $Z_o := Z \setminus (\sigma_2 E(\sigma_1) \cup \sigma_2 E(\sigma_2) \cup D)$ contains (infinitely many) generic fibres of f. That contradicts the Kobayashi hyperbolicity of Z_o and proves that the genus of C is 0 or 1.

(iii) According to Bogomolov-Katzarkov's [4], for any projective surface X' there is a blow-up $\sigma : S \to X'$ at finitely many points, so that the resulting surface S

admits a fibration $f : S \to C$ over a curve C. By (ii), the base C of f is an elliptic or a rational curve.

$\square$

Corollary 5. *If the toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ of a torsion free $\mathbb{B}/\Gamma$ is of Kodaira dimension $\kappa(X') \leq 1$, then the irregularity of X' is $q(X') = h^{0,1}(X') \leq 2$. The equality $q(X') = 2$ occurs if and only if the minimal model X of X' is an abelian surface.*

Proof. By Momot's [20], if $\kappa(X') \leq 0$ and $q(X') = h^{0,1}(X') \geq 1$ then $q(X) = 2$ and X is an abelian surface. We have to prove that the torsion free toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ of Kodaira dimension $\kappa(X') = 1$ have irregularity $q(X') \leq 1$. To this end, let us recall that the Albanese variety of a compact Kähler manifold M is the compact complex torus $Alb(M) := H^0(M, \Omega^1_M)^*/H_1(M, \mathbb{Z})$, where Ω^1_M stands for the sheaf of the holomorphic differential 1-forms on M. Note that $Alb(M)$ is of dimension $\dim_{\mathbb{C}} Alb(M) = h^{0,1}(M)$. For a compact complex curve C, the Albanese variety $Alb(C) = Jac(C)$ coincides with the Jacobian. The correspondence

$$p \mapsto (\omega \mapsto \int_{p_o}^{p} \omega) \quad \text{for} \quad \forall p \in M, \quad \forall \omega \in H^0(M, \Omega^1_M)$$

induces the Albanese map $alb_M : M \to Alb(M)$. By Beauville's [3], if $f : X \to C$ is a fibration of a smooth minimal surface X with elliptic generic fibre F then either $Alb(X) = Jac(C)$ or there is an exact sequence

$$0 \longrightarrow F' \longrightarrow Alb(X) \longrightarrow Jac(C) \longrightarrow 0 \qquad (5)$$

of abelian varieties for an elliptic curve F', isogeneous to F. By Lemma 4 (ii), the minimal models X of the torsion free toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ with $\kappa(X') = 1$ are elliptic fibrations $f : X \to C$ with a rational or an elliptic base C. The Jacobian of a rational curve C is a point, due to its simply connectedness. The irregularity of the elliptic curve F' equals the genus $q(F') = g(F') = 1$, so that the elliptic fibrations $f : X \to C$ with rational base have $q(X) = \dim_{\mathbb{C}} Alb(X) = 0$ or 1. The Jacobian $Jac(C) = C$ of an elliptic curve C coincides with C. Therefore the elliptic fibrations $f : X \to C$ with elliptic base C are of irregularity $q(X) = 1$ or 2, as far as the Albanese varieties $Alb(X)$, subject to (5) are of complex dimension $q(X) = \dim_{\mathbb{C}} Alb(X) = \dim_{\mathbb{C}} F' + \dim_{\mathbb{C}} C = 2$.

There remains to be shown the non-existence of a minimal model X of a torsion free toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ with $\kappa(X) = 1$, $q(X) = 2$ and an elliptic fibration $f : X \to C$ over an elliptic curve C. Assume the opposite and consider a torsion free $X' = (\mathbb{B}^2/\Gamma)'$, whose minimal model X of $\kappa(X) = 1$ and $q(X) = \dim_{\mathbb{C}} Alb(X) = 2$ is an elliptic fibration $f : X \to C$ with an elliptic base C. By the universal property of the Albanese variety Alb (X), the surjective morphism

$f : X \to C$ onto the compact complex torus C admits a factorization

$$\begin{tikzcd} X \arrow[r, "alb_X"] \arrow[d, "f"'] & Alb(X) \arrow[dl, "f_o"] \\ C & \end{tikzcd}$$

through the Albanese map alb_X of X and a surjective morphism $f_o : Alb(X) \to C$ of compact complex tori. Therefore all the fibres $f_o^{-1}(p)$, $p \in C$ of f_o are compact complex tori of $\dim_{\mathbb{C}} f_o^{-1}(p) = \dim_{\mathbb{C}} Alb(X) - \dim_{\mathbb{C}} C = 1$, i.e., elliptic curves. Now, in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "alb_X"] \arrow[d, "f"'] & alb_X(X) \arrow[dl, "f_o"] \\ C & \end{tikzcd}$$

of surjective morphisms, the fibres $f_o^{-1}(p)$, $p \in C$ of f_o are 1-dimensional and the image C of f_o is 1-dimensional. Thus, the image $alb_X(X)$ of the irreducible projective surface X under its Albanese map alb_X is a closed irreducible subvariety of $Alb(X)$ of $\dim_{\mathbb{C}} alb_X(X) = 2$ and $alb_X(X) = Alb(X)$. Now, the surjective morphism $alb_X : X \to Alb(X)$ is a finite covering of compact complex surfaces. For an arbitrary $p \in C$ the elliptic fibre $f_o^{-1}(p)$ of f_o is a finite quotient of the fibre $f^{-1}(p)$ of f, due to the presence of a commutative diagram

$$\begin{tikzcd} f^{-1}(p) \arrow[r, "alb_X"] \arrow[d, "f"'] & alb_X(f^{-1}(p)) = f_o^{-1}(p) \arrow[dl, "f_o"] \\ C & \end{tikzcd}$$

of surjective morphisms. Therefore the irreducible components of $f^{-1}(p)$ are of genus ≥ 1 and the elliptic fibration $f : X \to C$ has not singular fibres. Note that the finite ramified coverings of the elliptic curves $f_o^{-1}(p)$ are of genus ≥ 2. Therefore, the finite coverings $alb_X : f^{-1}(p) \to alb_X(f^{-1}(p)) = f_o^{-1}(p)$ of elliptic curves are unramified for $\forall p \in C$ and the Albanese map $alb_X : X \to Alb(X)$ is a finite unramified covering. As a result, X is of Kodaira dimension $\kappa(X) = \kappa(Alb(X)) = 0$, contrary to the assumption $\kappa(X) = 1$. The contradiction justifies the non-existence of torsion free toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ of $\kappa(X') = 1$ and $q(X') = 2$, whose minimal models X are elliptic fibrations $f : X \to C$ over elliptic curves C.

$\square$

Let $\eta : X' = (\mathbb{B}^2/\Gamma) \to \widehat{\mathbb{B}^2/\Gamma} = (\mathbb{B}^2/\Gamma) \cup (\partial_{\Gamma}\mathbb{B}^2/\Gamma)$ be the contraction of the toroidal compactifying divisor $T = (\mathbb{B}^2/\Gamma)' \setminus (\mathbb{B}^2/\Gamma)$ of a torsion free $\mathbb{B}^2/\Gamma$ to the

Baily-Borel compactification $\widehat{\mathbb{B}^2/\Gamma}$ of $\mathbb{B}^2/\Gamma$ (cf.[2]). One can view the Γ-modular forms of weight $n \in \mathbb{N}$ as global holomorphic sections of the pluri-canonical bundles $\mathcal{K}_{\mathbb{B}^2/\Gamma}^{\otimes n}$ of $\mathbb{B}^2/\Gamma$. Hemperly has shown in [8] that the pull back $\eta^* H^0(\mathbb{B}^2/\Gamma, \mathcal{K}_{\mathbb{B}^2/\Gamma}^{\otimes n}) = H^0(X', \mathcal{K}_{X'}(T)^{\otimes n})$ coincides with the space of the global holomorphic sections of the logarithmic-pluri-canonical bundles $\mathcal{K}_{X'}(T)^{\otimes n}$ of X'. The Γ-modular forms, which vanish on all the cusps of $\mathbb{B}^2/\Gamma$ are called cuspidal. In [14] Holzapfel specifies that the cuspidal Γ-modular forms of weight $n \in \mathbb{N}$ pull back to $H^0(X', \mathcal{K}_{X'}^{\otimes n} \otimes \mathcal{O}_{X'}(T)^{\otimes (n-1)})$ by η. The space of the cuspidal Γ-modular forms of weight 1 is 1-dimensional, so that $H^0(X', \mathcal{K}_{X'}) \simeq \mathbb{C}$. In such a way, we obtain the following

Corollary 6. *The toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ of a torsion free $\mathbb{B}^2/\Gamma$ has geometric genus $p_g(X') = h^{2,0}(X') = 1$.*

3 Local complex hyperbolic surfaces of minimal volume

Let $\mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic number field with integers ring $\mathcal{O}_{-d}$. Denote by $PU_{n,1}(\mathcal{O}_{-d})$ the projective unitary group with entries from $\mathcal{O}_{-d}$. The lattices Γ of $PU_{n,1}$, which are commensurable with $PU_{n,1}(\mathcal{O}_{-d})$ are called Picard modular over $\mathbb{Q}(\sqrt{-d})$. Any arithmetic lattice of $PU_{n,1}$ is Picard modular over $\mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{N}$. In [6], Emery and Stover express the minimal volume $v_{-d,n}$ of a quotient $PU_{n,1}/\Gamma$ by a non-uniform Picard modular lattice $\Gamma < PU_{n,1}$ over $\mathbb{Q}(\sqrt{-d})$ by the L-function $L_{-d} = \frac{\zeta_{-d}}{\zeta}$, associated with Dedekind zeta function ζ_{-d} of $\mathbb{Q}(\sqrt{-d})$ and the Riemann zeta function ζ. They estimate the number of the isomorphism classes of the Picard modular lattices $\Gamma < PU_{n,1}$ over $\mathbb{Q}(\sqrt{-d})$ with minimal volume $\mathrm{vol}(PU_{n,1}/\Gamma) = v_{-d,n}$, For all $n \geq 2$, the non-uniform arithmetic lattices $\Gamma < PU_{n,1}$ of smallest $\mathrm{vol}(PU_{n,1}/\Gamma) = v_{-d,n}$ are shown to be Picard modular over $\mathbb{Q}(\sqrt{-3})$. For an even $n = 2k$, there are exactly two isomorphism classes of non-uniform lattices of $PU_{2k,1}$ with minimal co-volume $v_{-3,2k}$. For an odd $n = 2k+1 \not\equiv 7 (\mathrm{mod}\ 8)$ there is a unique isomorphism class of non-uniform lattices of $PU_{2k+1,1}$ with minimal co-volume $v_{-3,2k+1}$.

The previous work [24] of Stover establishes that the minimal volume of a non-compact arithmetic quotient $\mathbb{B}^2/\Gamma$ is $\frac{\pi^2}{27}$. According to Parker's [21], the minimal volume of a torsion free non-compact discrete quotient $\mathbb{B}^2/\Gamma$ is $\frac{8\pi^2}{3}$. Stover shows in [24] that any torsion free arithmetic $\mathbb{B}^2/\Gamma$ of $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ covers (at least) one of the two non-isomorphic torsion Picard modular surfaces $\mathbb{B}^2/\Gamma_{-3}$ or $\mathbb{B}^2/\Gamma'_{-3}$ with minimal volume $\mathrm{vol}(\mathbb{B}^2/\Gamma_{-3}) = \mathrm{vol}(\mathbb{B}^2/\Gamma'_{-3}) = \frac{\pi^2}{27}$. He provides a complete list of representatives of the isomorphism classes of torsion free $\Gamma < \Gamma_{-3} \cap \Gamma'_{-3}$ with $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$. The non-arithmetic non-compact discrete quotients $\mathbb{B}^2/\Gamma$ are not expected to be of minimal volume $\frac{\pi^2}{27}$.

In [18] the author establishes that any admissible value $\frac{8\pi^2}{3}n \in \frac{8\pi^2}{3}\mathbb{N}$ for the volume of a quotient $\mathbb{B}^2/\Gamma$ by a torsion free lattice $\Gamma < SU_{2,1}$ is attained by a Picard modular Γ_n over $\mathbb{Q}(\sqrt{-3})$, whose associated toroidal compactification $(\mathbb{B}^2/\Gamma)'$ is birational to an abelian surface. The next proposition discusses the Kodaira-Enriques classification type of the minimal model X of a torsion free $X' = (\mathbb{B}^2/\Gamma)'$ of minimal

volume $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$. It derives some estimates for the number of cusps $h(X')$ of $\mathbb{B}^2/\Gamma$.

Proposition 7. *Let X be the minimal model of a torsion free toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ of minimal volume*

$$\mathrm{vol}(X') = \mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}.$$

Then the Kodaira-Enriques classification type of X, the Euler number $e(X)$ of X, the self-intersection number $K_X^2 \in \mathbb{Z}$ of the canonical divisor K_X, the number $h(X')$ of the cusps of $\mathbb{B}^2/\Gamma$ and the number $s(X')$ of the smooth rational (-1)-curves on X' are among the ones, listed in the following table:

Type	$e(X)$	$h(X')$	K_X^2	$s(X')$
Abelian surface X	$e(X) = 0$	$h(X') = 4$	$K_X^2 = 0$	$s(X') = 1$
$\kappa(X) = 1$	$e(X) = 0$	$3 \leq h(X') \leq 4$	$K_X^2 = 0$	$s(X') = 1$
$X' = X$ of $\kappa(X) = 1$	$e(X) = 1$	$1 \leq h(X') \leq 3$	$K_X^2 = 0$	$s(X') = 0$
$X' = X$ of general type	$e(X) = 1$	$1 \leq h(X') \leq 2$	$K_X^2 = 1$	$s(X') = 0$

If $f : X \to C$ is an elliptic fibration with $\kappa(X) = 1$ and $e(X) = 0$ then f has no singular fibres.

The elliptic fibrations $f : X \to C$ of $\kappa(X) = 1$ and $e(X) = 1$ have exactly one singular fibre, which is an irreducible rational curve with a double point.

Proof. By results of Hirzebruch from [10], [11], $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3} e(\mathbb{B}^2/\Gamma)$ for the Euler number $e(\mathbb{B}^2/\Gamma) \in \mathbb{N}$ of $\mathbb{B}^2/\Gamma$. Thus, a torsion free $\mathbb{B}^2/\Gamma$ has minimal volume $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ exactly when the Euler number $e(\mathbb{B}^2/\Gamma) = 1$. The toroidal compactifying divisor consists of disjoint smooth irreducible elliptic curves, so that $e(X') = e(\mathbb{B}^2/\Gamma) = 1$. If X' contains $s(X') \in \mathbb{Z}^{\geq 0}$ smooth rational (-1)-curves then $1 = e(X') = e(X) + s(X')$. The minimal surface X has non-negative Euler number $e(X) = 1 - s(X') \geq 0$, so that $s(X') \in \{0, 1\}$ and $e(X) \in \{0, 1\}$.

If $\kappa(X') \leq 0$ and $q(X') = h^{0,1}(X') = 0$ then the minimal surface X is rational, K3 or an Enriques surface. The minimal K3 surfaces have Euler number 24 and the minimal Enriques surfaces have Euler number 12. The minimal rational surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and Hirzebruch surfaces Σ_n with $n \geq 2$. The Euler numbers $e(\mathbb{P}^2) = 3$, $e(\mathbb{P}^1 \times \mathbb{P}^1) = e(\Sigma_n) = 4$ for $\forall n \in \mathbb{N}$ are strictly greater than 1. More precisely, Σ_1 is the blow-up of $\mathbb{P}^2$ at one point. For any $n \geq 2$ the surface Σ_n is obtained from Σ_{n-1} by blow-up of a point and contraction of a smooth rational curve. Therefore $e(\Sigma_n) = e(\Sigma_{n-1})$. The above considerations show that if $\kappa(X') \leq 0$ then $q(X') \geq 1$. In such a case, Momot's [20] specifies that X is an abelian surface. Bearing in mind that an abelian surface X has $e(X) = 0$ and $K_X^2 = 0$, one concludes that $X' = (\mathbb{B}^2/\Gamma)'$ contains $s(X') = 1$ smooth rational curve.

In order to recall Holzapfel's proportionality condition on a torsion free toroidal compactification $Y' = (\mathbb{B}^2/\Gamma_o)'$, let us consider the blow-down $\xi : Y' \to Y$ of the smooth rational (-1)-curves on Y' and the image $D = \xi(T)$ of the toroidal

compactifying divisor $(\mathbb{B}^2/\Gamma_o)' \setminus (\mathbb{B}^2/\Gamma_o) = T = \sum_{i=1}^{h(X')} T_i$ under ξ. The divisor $D = \sum_{i=1}^{h(Y')} D_i$ has smooth elliptic irreducible components $D_i = \xi(T_i)$ and the singular locus $D^{\text{sing}} = \sum_{1\leq i<j\leq h(Y')} D_i \cap D_j$ of D consists of the intersection points of the different components. The toroidal compactification $Y' = (\mathbb{B}^2/\Gamma_o)'$ is the blow-up of Y of D^{sing}, so that the number $s(Y')$ of the smooth rational (-1)-curves on Y' coincides with the cardinality $|D^{\text{sing}}| = s(Y')$ of the singular locus D^{sing} of D. Holzapfel shows in [13] that the minimal model Y of a torsion free toroidal compactification $Y' = (\mathbb{B}^2/\Gamma_o)'$ and the divisor $D \subset Y$ satisfy the proportionality condition

$$3e(Y) - K_Y^2 = \sum_{i=1}^{h(Y')} K_Y.D_i + \sum_{i=1}^{h(Y')} |D_i \cap D^{\text{sing}}| - 4s(Y'). \tag{6}$$

In the case of an abelian minimal model Y, the canonical bundle $\mathcal{K}_Y = \mathcal{O}_Y$ is trivial and $e(Y) = 0$, $K_Y^2 = 0$. Holzapfel's proportionality reduces to

$$\sum_{i=1}^{h(Y')} |D_i \cap D^{\text{sing}}| = 4s(Y'). \tag{7}$$

If ξ blows up $s_i(Y') = |D_i \cap D^{\text{sing}}|$ points on D_i then $T_i^2 = D_i^2 - s_i(Y')$. According to the adjunction formula

$$0 = -e(D_i) = D_i(D_i + K_Y) = D_i^2$$

for the elliptic curves D_i on the abelian surface Y and the contractibility condition $T_i^2 < 0$ on $T_i \subset Y' = (\mathbb{B}^2/\Gamma_o)'$, one has $s_i(Y') \in \mathbb{N}$ for all $1 \leq i \leq h(Y')$. As a result, (7) implies that $h(Y') \leq 4s(Y') \leq h(Y')s(Y')$. In the special case of $s(X') = 1$ there follows $h(X') = 4s(X') = 4$.

If $\kappa(X') = 1$ then the minimal model X of $X' = (\mathbb{B}^2/\Gamma)'$ is an elliptic fibration $f : X \to C$. By Lemma 4 (ii), the base C of f is a rational or an elliptic curve. Denoting by $F_1, \ldots, F_n$ the singular fibres of f, one expresses the Euler number in the form $e(X) = \sum_{j=1}^{n} e(F_j)$. If F_j is an irreducible rational curve with a double point then $e(F_j) = 1$. We claim that all the other types of singular fibres have Euler number ≥ 2. In terms of Kodaira symbols of the singular fibres, one has

$$e(III) = 3, \quad e(IV) = 4, \quad e(II^*) = 10, \quad e(III^*) = 9, \quad e(IV^*) = 8,$$

$$e(I_n^*) = n + 6 \quad \text{for} \quad \forall n \geq 0.$$

These are computed by observing that for any of the listed singular fibres F_j there is a permutation $F_{j,1}, \ldots, F_{j,k}$ of the smooth irreducible rational components $F_{j,i}$ of F_j, such that $F_{j,i}$ intersects $\cup_{s=1}^{i-1} F_{j,s}$ in a single point for all $2 \leq i \leq k$. If F_j has k irreducible components then the Euler number $e(F_j) = e(\cup_{i=1}^{k} F_{j,k}) = k + 1$.

For the singular fibres of type I_n with $n \geq 2$, there is a permutation $I_{n,1}, \dots, I_{n,n}$ of the smooth irreducible rational components $I_{n,j}$ of I_n, such that $I_{n,i}$ intersects $\cup_{s=1}^{i-1} I_{n,s}$ in a single point for $2 \leq i \leq n-1$ and $I_{n,n}$ intersects $\cup_{s=1}^{n-1} I_{n,s}$ in two points. Therefore $e(I_n) = n$. A torsion free $X' = (\mathbb{B}^2/\Gamma)'$ of Kodaira dimension $\kappa(X') = 1$ has minimal model X with $e(X) = 0$ if and only if the elliptic fibration $f : X \to C$ has no singular fibres and X' is the blow-up of X at one point. The case of $\kappa(X') = 1$ and $e(X) = 1$ occurs exactly when $f : X \to C$ has one singular fibre, which is an irreducible rational curve with a double point and $X' = X$ is a minimal surface.

The canonical divisor K_X of a minimal surface X of Kodaira dimension $\kappa(X) = 1$ has vanishing self-intersection number $K_X^2 = 0$. By the adjunction formula

$$0 = -e(D_i) = (D_i + K_X)D_i = D_i^2 + K_X.D_i$$

for the elliptic curves $D_i \subset X$ and $T_i^2 = D_i^2 - |D_i \cap D^{\mathrm{sing}}|$, Holzapfel's proportionality takes the form

$$3e(X) + 4s(X') = \sum_{i=1}^{h(X')} (-T_i^2)$$

with $-T_i^2 \geq 1$. In the case of $e(X) = 0$ and $s(X') = 1$, there follows $4 = \sum_{i=1}^{h(X')} (-T_i^2) \geq h(X')$. By Momot's [20], the exceptional divisor $E \simeq \mathbb{P}^1(\mathbb{C})$ of the blow-up $\xi : X' \to X$ of X at the unique singular point of D intersects $T = (\mathbb{B}^2/\Gamma)' \setminus (\mathbb{B}^2/\Gamma)$ in at least three points, due to Kobayashi hyperbolicity of $\mathbb{B}^2/\Gamma$. The morphism ξ of degree 1 restricts to isomorphisms $\xi : T_i \to D_i = \xi(T_i)$ of the irreducible components T_i of T, so that E intersects each T_i in at most one point. Thus, E intersects at least three different irreducible components T_1, T_2, T_3 of T and the total number $h(X')$ of the irreducible components of T is $h(X') \geq 3$.

If the torsion free toroidal compactification $Y' = (\mathbb{B}^2/\Gamma_o)' = Y$ is a minimal surface then $s(Y') = 0$ and $|D_i \cap D^{\mathrm{sing}}| = 0$ for $\forall 1 \leq i \leq h(Y')$. Moreover, $T_i = D_i$ requires $D_i^2 \leq -1$ and Holzapfel's proportionality reduces to

$$3e(Y) - K_Y^2 = \sum_{i=1}^{h(Y')} (-D_i^2) \geq h(Y'). \tag{8}$$

In particular, for $X' = X$ of $\kappa(X) = 1$ and $e(X) = 1$, one concludes that $h(X') \leq 3$. According to Hersonsky and Paulin's [9], the minimal volume of a compact torsion free discrete quotient $\mathbb{B}^2/\Gamma$ is $\mathrm{vol}(\mathbb{B}^2/\Gamma) = 8\pi^2$, so that $\mathbb{B}^2/\Gamma$ of $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ is to be non-compact. In other words, $\mathbb{B}^2/\Gamma$ has to have at least one cusp, $h(X') \geq 1$.

If $X' = (\mathbb{B}^2/\Gamma)'$ is of general type then $e(X) \geq 1$. Therefore $e(\mathbb{B}^2/\Gamma) = 1$ if and only if $X' = X$ is a minimal surface with $e(X) = 1$. Bogomolov-Miyaoka-Yau's inequality asserts that $K_X^2 \leq 3e(X) = 3$ with equality $K_X^2 = 3$ exactly when $X = \mathbb{B}^2/\Gamma_0$ is a compact torsion free ball quotient. The equality $K_X^2 = 2e(X) = 2$ holds for compact torsion free quotients $\mathbb{B}^1 \times \mathbb{B}^1/\Gamma_1$ of the bi-disc $\mathbb{B}^1 \times \mathbb{B}^1$. Bearing in mind the Kobayashi hyperbolicity of $\mathbb{B}^2/\Gamma_0$ and $\mathbb{B}^1 \times \mathbb{B}^1/\Gamma_1$, one concludes that

$K_X^2 \le 1$. The minimal surface X of general type has $K_X^2 > 0$, whereas $K_X^2 = 1$. Holzapfel's proportionality for the case under consideration reads as

$$2 = 3e(X) - K_X^2 = \sum_{i=1}^{h(X')} (-D_i^2)$$

with $-D_i^2 = -T_i^2 \ge 1$. Therefore $h(X') \le \sum_{i=1}^{h(X')} (-D_i^2) = 2$. The non-compact $\mathbb{B}^2/\Gamma$ has $h(X') \ge 1$ cusps.

$\square$

By Stover's [24] , there are two non-isomorphic Picard modular lattices Γ_{-3}, Γ'_{-3} of $SU_{2,1}$ over $\mathbb{Q}(\sqrt{-3})$ with minimal $\mathrm{vol}(\mathbb{B}^2/\Gamma_{-3}) = \mathrm{vol}(\mathbb{B}^2/\Gamma'_{-3}) = \frac{\pi^2}{27}$. Any arithmetic torsion free lattice $\Gamma < SU_{2,1}$ with minimal co-volume $\mathrm{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ is contained in Γ_{-3} or in Γ'_{-3}. The appendix of [24] provides a complete list of the pairwise non-isomorphic arithmetic torsion free $\Gamma_j < SU_{2,1}$, $1 \le j \le 8$ with $\mathrm{vol}(\mathbb{B}^2/\Gamma_j) = \frac{8\pi^2}{3}$, which are contained in $\Gamma_{-3} \cap \Gamma'_{-3}$. Stover provides generators of $\Gamma_j < SU_{2,1}(\mathcal{O}_{-3})$ in terms of the two generators of $SU_{2,1}(\mathcal{O}_{-3})$, found by Falbel-Parker in [7]. He computes the homology group $H_1(\mathbb{B}^2/\Gamma_j, \mathbb{Z})$ and the number $h(\mathbb{B}^2/\Gamma_j) = h((\mathbb{B}^2/\Gamma_j)')$ of the cusps of $\mathbb{B}^2/\Gamma_j$ for all $1 \le j \le 8$. According to Corollary 3, the toroidal compactifications $X'_j = (\mathbb{B}^2/\Gamma_j)'$ have irregularity

$$q(X'_j) = h^{0,1}(X'_j) = \frac{1}{2}\mathrm{rk} H^1(X'_j, \mathbb{Z}) = \frac{1}{2} H_1(\mathbb{B}^2/\Gamma_j, \mathbb{Z}).$$

Stover's examples $\mathbb{B}^2/\Gamma_j$ with $1 \le j \le 3$ have $h(X'_j) = 4$ cusps, while $\mathbb{B}^2/\Gamma_j$ with $4 \le j \le 8$ have $h(X'_j) = 2$ cusps. By Proposition 7, the torsion free $X'_j = (\mathbb{B}^2/\Gamma_j)'$ of minimal volume $\mathrm{vol}(X'_j) = \frac{8\pi^2}{3}$ with $h(X'_j) = 4$ cusps are of Kodaira dimension $\kappa(X'_j) \le 1$. Due to $q(X'_1) = q(X'_2) = 0$, the minimal models X_j of X'_j with $1 \le j \le 2$ are elliptic fibrations $X_j \to \mathbb{P}^1(\mathbb{C})$ with rational base and without singular fibres. According to Corollary 5, the minimal model X_3 of $X'_3 = (\mathbb{B}^2/\Gamma_3)'$ with $q(X'_3) = 2$ is an abelian surface. There is an example $X'_{\mathrm{Hir}} = (\mathbb{B}^2/\Gamma_{\mathrm{Hir}})'$ of Hirzebruch for a torsion free Picard modular surface with abelian minimal model and $\mathrm{vol}(\mathbb{B}^2/\Gamma_{\mathrm{Hir}}) = \frac{8\pi^2}{3}$. More precisely, in [12] Hirzebruch constructs an infinite series $\{Z_n\}_{n=1}^{\infty}$ of minimal surfaces of general type with $\lim\limits_{n\to\infty} \frac{K_{Z_n}^2}{e(Z_n)} = 3$. The surfaces Z_n are birational to branched covers of the abelian surface $A_{-3} = (\mathbb{C}/\mathcal{O}_{-3}) \times (\mathbb{C}/\mathcal{O}_{-3})$, ramified over four elliptic curves $D_1, \dots, D_4 \subset A_{-3}$, intersecting in the origin alone. Holzapfel shows in [15] that the blow-up $X'_{\mathrm{Hir}} = (\mathbb{B}^2/\Gamma_{\mathrm{Hir}})'$ of A_{-3} at the origin is a torsion free Picard modular toroidal compactification over $\mathbb{Q}(\sqrt{-3})$. If there are no co-abelian torsion free $\mathbb{B}^2/\Gamma_o$ of $\mathrm{vol}(\mathbb{B}^2/\Gamma_o) = \frac{8\pi^2}{3}$, which cover exactly one of $\mathbb{B}^2/\Gamma_{-3}$ or $\mathbb{B}^2/\Gamma'_{-3}$, then Stover's example $X'_3 = (\mathbb{B}^2/\Gamma_{\mathrm{Hir}})'$ coincides with the one of Hirzebruch.

Stover's $X'_j = (\mathbb{B}^2/\Gamma_j)'$ with $h(X'_j) = 2$ for $4 \le j \le 8$ are minimal surfaces $X'_j = X_j$ of $\kappa(X_j) \ge 1$, according to Proposition 7. The torsion free lattice Γ_6 has 2 generators and the minimal surface $X'_6 = X_6$ has vanishing irregularity $q('X'_6) = 0$. Stover's lattices Γ_j with $j \in \{4, 5, 7, 8\}$ have 3 generators and the surfaces $X'_j = X_j$ are of irregularity $q(X'_j) = 1$.

4 An application of residual finiteness of lattices in $SU_{n,1}$

The fundamental group $\pi_1(\mathbb{B}/\Gamma)' = \Gamma/\Gamma^U$ of a torsion free toroidal compactification is a quotient group of the residually finite lattice Γ of $SU_{n,1}$. Note that $\pi_1(X')$ is not supposed to be residually finite, as far as the intersection of the finite index subgroups of Γ, containing Γ^U could contain strictly Γ^U, regardless of the fact that all the finite index subgroups of Γ intersect in the identity alone.

In order to apply the residual finiteness of the lattices $\Gamma < SU_{n,1}$, we proceed with some obvious properties of the finite coverings of toroidal compactifications of torsion free local complex hyperbolic spaces.

Definition 8. *The finite surjective holomorphic map*

$$f : X_2' = (\mathbb{B}^n/\Gamma_2)' \longrightarrow X_1' = (\mathbb{B}^n/\Gamma_1)'$$

is a finite covering of toroidal compactifications of torsion free local complex hyperbolic spaces if $f^{-1}(\mathbb{B}^n/\Gamma_1) = \mathbb{B}^n/\Gamma_2$, $f : \mathbb{B}^n/\Gamma_2 \to \mathbb{B}^n/\Gamma_1$ is unramified and the ramification index of f is constant over any irreducible component $T(2)_i$ of the toroidal compactifying divisor $T(2) = (\mathbb{B}^n/\Gamma_2)' \setminus (\mathbb{B}^n/\Gamma_2)$ of X_2'.

Towards a characterization of the finite coverings of the toroidal compactifications of torsion free local complex hyperbolic spaces, recall that the ineffective kernel of the $SU_{n,1}$-action on $\mathbb{B}^n$ is the center

$$Z(SU_{n,1}) = \left\{ e^{\frac{2\pi i}{n+1}} I_{n+1} \ \Big| \ 0 \leq k \leq n \right\}$$

of $SU_{n,1}$. Denote by $\mathbb{P} : SU_{n,1} \to PU_{n,1} = SU_{n,1}/Z(SU_{n,1})$ the natural epimorphism and note that $\mathbb{B}^n/\Gamma = \mathbb{B}^n/\mathbb{P}(\Gamma)$ for any lattice $\Gamma < SU_{n,1}$.

Lemma 9. *(i) If Γ is a torsion free lattice of $SU_{n,1}$ and Γ_o is a subgroup of Γ of finite index, then $\partial_\Gamma \mathbb{B}^n = \partial_{\Gamma_o}\mathbb{B}^n$. For any $g \in SU_{n,1}$ and any subgroup Γ_o of Γ of finite index, the automorphism $g : \mathbb{B}^n \to \mathbb{B}^n$ of the ball $\mathbb{B}^n$ induces a finite covering*

$$\varphi : (\mathbb{B}^n/\mathbb{P}(g)^{-1}\mathbb{P}(\Gamma_o)\mathbb{P}(g))' \longrightarrow (\mathbb{B}^n/\mathbb{P}(\Gamma))'$$

of toroidal compactifications of torsion free local complex hyperbolic spaces.

(ii) Suppose that Γ_1 is a torsion free lattice of $SU_{n,1}$ and Γ_2 is a normal subgroup of Γ_1 of finite index. For any $p \in \partial_{\Gamma_1}\mathbb{B}^n$ let $W(p)$ be the unipotent radical of $Stab(p) < SU_{n,1}$, $\Gamma_j^W(p) := \Gamma_j \cap W(p)$, $U(p)$ be the commutant of $W(p)$, $\Gamma_j^U(p) := \Gamma_j \cap U(p)$, $\zeta_{\Gamma_j}(p) \in \partial_{\Gamma_j}\mathbb{B}^n/\Gamma_j$ be the Γ_j-cusp, associated with p and $T(\zeta_j(p))$ be the irreducible component of $(\mathbb{B}^n/\Gamma_j)' \setminus (\mathbb{B}^n/\Gamma_j)$, corresponding to $\zeta_{\Gamma_j}(p)$. Then the Γ_1/Γ_2-covering $\varphi : X_2' = (\mathbb{B}^n/\Gamma_2)' \to X_1' = (\mathbb{B}^n/\Gamma_1)'$, induced by $\mathrm{Id} : \mathbb{B}^n \to \mathbb{B}^n$ restricts to a covering $\varphi : T(\zeta_{\Gamma_2}(p)) \to T(\zeta_{\Gamma_1}(p))$, whose all fibres are acted effectively by the group $\Gamma_1^W(p)/\Gamma_1^U(p)\Gamma_2^W(p)$.

(iii) Any finite covering $\varphi : (\mathbb{B}^n/\Gamma_2)' \to (\mathbb{B}^n/\Gamma_1)'$ of torsion free toroidal compactifications lifts to an automorphism $g : \mathbb{B}^n \to \mathbb{B}^n$ of the ball $\mathbb{B}^n$, such that $\mathbb{P}(g\Gamma_2 g^{-1})$ is a subgroup of $\mathbb{P}(\Gamma_1)$ of finite index $[\mathbb{P}(\Gamma_1) : \mathbb{P}(g\Gamma_2 g^{-1})] = \deg(\varphi)$ and

$$\varphi : (\mathbb{B}^n/\Gamma_2)' = (\mathbb{B}^n/\mathbb{P}(g)^{-1}\mathbb{P}(g\Gamma_2 g^{-1})\mathbb{P}(g))' \longrightarrow (\mathbb{B}^n/\mathbb{P}(\Gamma_1))'.$$

Proof. (i) An arbitrary automorphism $g : \mathbb{B}^n \to \mathbb{B}^n$ is equivariant with respect to the Γ_o-action on the target and the $g^{-1}\Gamma_o g$-action on the source. Therefore $g \in Aut(\mathbb{B}^n) = SU_{n,1}$ induces an isomorphism

$$f : \mathbb{B}^n/g^{-1}\Gamma_o g \longrightarrow \mathbb{B}^n/\Gamma_o,$$

$$f(\zeta_{g^{-1}\Gamma_o g}(z)) = \zeta_{\Gamma_o}(gz) \quad \text{for} \quad \forall z \in \mathbb{B}^n.$$

Bearing in mind that Γ_o is a subgroup of Γ, one obtains a morphism

$$\varphi : \mathbb{B}^n/g^{-1}\Gamma_o g \longrightarrow \mathbb{B}^n/\Gamma,$$

$$\varphi(\zeta_{g^{-1}\Gamma_o g}(z)) = \zeta_{\Gamma}(gz) \quad \text{for} \quad \forall z \in \mathbb{B}^n.$$

The induced homomorphism $\varphi_* : \pi_1(\mathbb{B}^n/g^{-1}\Gamma_o g) = g^{-1}\Gamma_o g \to \Gamma = \pi_1(\mathbb{B}^n/\Gamma)$, $\varphi_*(g^{-1}\gamma_o g) = \gamma_o$ for $\forall \gamma_o \in \Gamma_o$ of the fundamental groups is an embedding with $\mathrm{im}(\varphi_*) = \Gamma_o$ of index $[\Gamma : \Gamma_o] = m$. As far as Γ and $g^{-1}\Gamma_o g$ are torsion free, φ is an unramified covering of degree $\deg(\varphi) = m$.

For the subgroup $\Gamma_o < \Gamma$ of finite index $[\Gamma : \Gamma_o] = m$, we claim that $\partial_{\Gamma}\mathbb{B}^n = \partial_{\Gamma_o}\mathbb{B}^n$. Namely, $p \in \partial_{\Gamma}\mathbb{B}^n$ is a Γ-rational boundary point if and only if the unipotent radical $W(p)$ of $Stab(p) < SU_{n,1}$ intersects Γ is a lattice $\Gamma^W(p) := \Gamma \cap W(p)$ of $W(p)$. Bearing in mind that $\Gamma^W(p)/\Gamma_o^W(p) \simeq \Gamma^W(p)\Gamma_o/\Gamma_o$ is a subset of the finite coset space Γ/Γ_o, one observes that $W(p)/\Gamma_o^W(p) \to W(p)/\Gamma^W(p)$ is a finite covering. Therefore $W(p)/\Gamma^W(p)$ has finite invariant measure exactly when $W(p)/\Gamma_o^W(p)$ has finite invariant measure and $p \in \partial_{\Gamma}\mathbb{B}^n$ is equivalent to $p \in \partial_{\Gamma_o}\mathbb{B}^n$.

For an arbitrary $h \in SU_{n,1}$ one has $\partial_{h^{-1}\Gamma_o h}\mathbb{B}^n = h^{-1}\partial_{\Gamma_o}\mathbb{B}^n$. More precisely, if $\gamma_o \in \Gamma_o$ is a parabolic element with unique fixed point $p \in \partial\mathbb{B}^n$, then $h^{-1}\gamma_o h \in h^{-1}\Gamma_o h$ is a parabolic element with unique fixed point $h^{-1}(p)$. Therefore $h^{-1}\partial_{\Gamma_o}\mathbb{B}^n \subseteq \partial_{h^{-1}\Gamma_o h}\mathbb{B}^n$. Replacing Γ_o by $h^{-1}\Gamma_o h$ and h by h^{-1}, one obtains $h\partial_{h^{-1}\Gamma_o h}\mathbb{B}^n \subseteq \partial_{\Gamma_o}\mathbb{B}^n$, whereas $h^{-1}\partial_{\Gamma_o}\mathbb{B}^n = \partial_{h^{-1}\Gamma_o h}\mathbb{B}^n$.

For an arbitrary cusp $\kappa \in \partial_{g^{-1}\Gamma_o g}\mathbb{B}^n/g^{-1}\Gamma_o g$, let us fix a boundary point $p \in \partial_{g^{-1}\Gamma_o g}\mathbb{B}^n$ with $\zeta_{g^{-1}\Gamma_o g}(p) = \kappa$ and note that $gp \in \partial_{\Gamma_o}\mathbb{B}^n = \partial_{\Gamma}\mathbb{B}^n$. The finite unramified covering $\varphi : \mathbb{B}^n/g^{-1}\Gamma_o g \to \mathbb{B}^n/\Gamma$ extends to an eventually ramified finite covering

$$\varphi : \partial_{g^{-1}\Gamma_o g}\mathbb{B}^n/g^{-1}\Gamma_o g \longrightarrow \partial_{\Gamma}\mathbb{B}^n,$$

$$\varphi(\kappa) = \varphi(\zeta_{g^{-1}\Gamma_o g}(p)) = \zeta_{\Gamma}(gp)$$

of the corresponding cusps. For sufficiently large $N \in \mathbb{N}$, fix a horoball neighborhood $\mathbb{B}^n(p, N)$ of $p \in \partial_{g^{-1}\Gamma_o g}\mathbb{B}^n$ on $\mathbb{B}^n$ and the neighborhood

$$V(\widehat{\zeta_{g^{-1}\Gamma_o g}(p)}, N) = (\mathbb{B}^n(p,N)/g^{-1}\Gamma_o g) \cup T(\zeta_{g^{-1}\Gamma_o g}(p))$$

of the compact complex torus $T(\zeta_{g^{-1}\Gamma_o g}(p)) \simeq T_o(p) := \mathbb{C}^{n-1}/\Lambda_o(p)$, $\Lambda_o(p) := (g^{-1}\Gamma_o g \cap W(p))/(g^{-1}\Gamma_o g \cap U(p))$ on $(\mathbb{B}^n/g^{-1}\Gamma_o g)'$. Consider the neighborhood

$$V(\widehat{\zeta_{\Gamma}(gp)}, N) = (\mathbb{B}^n(gp, N)/\Gamma) \cup T(\zeta_{\Gamma}(gp))$$

of $T(\zeta_\Gamma(gp)) \simeq T(gp) := \mathbb{C}^{n-1}/\Lambda(gp)$, $\Lambda(gp) := (\Gamma \cap W(gp))/(\Gamma \cap U(gp))$. In order to extend $\varphi : \mathbb{B}^n/g^{-1}\Gamma_o g \to \mathbb{B}^n/\Gamma$ to a finite covering

$$\varphi : (\mathbb{B}^n/g^{-1}\Gamma_o g) \cup V(\widehat{\zeta_{g^{-1}\Gamma_o g}(p)}, N) \longrightarrow (\mathbb{B}^n/\Gamma) \cup V(\widehat{\zeta_\Gamma(gp)}, N),$$

eventually ramified over $T(\zeta_\Gamma(gp)$, let us recall the isomorphisms

$$\widehat{Z_o(p,N)} := [\mathbb{B}^n(p,N)/g^{-1}\Gamma_o g \cap W(p)] \cup T_o(p) \longrightarrow V(\widehat{\zeta_{g^{-1}\Gamma_o g}(p)}, N),$$

$$\widehat{Z(gp,N)} := [\mathbb{B}^n(gp,N)/\Gamma \cap W(gp)] \cup T(gp) \longrightarrow V(\widehat{\zeta_\Gamma(gp)}, N)$$

from Lemma 1 and justify the existence of a finite covering

$$\varphi^W : \widehat{Z_o(p,N)} \longrightarrow \widehat{Z(gp,N)}.$$

More precisely, $g : \mathbb{B}^n \to \mathbb{B}^n$ is equivariant with respect to the action of $\Gamma \cap U(gp)$ and $\Gamma \cap W(gp)$ on the target and the action of $g^{-1}\Gamma_o g \cap U(p)$, respectively, $g^{-1}\Gamma_o g \cap W(p)$ on the source. Therefore g induces finite unramified coverings

$$\varphi^U : \mathbb{B}^n(p,N)/(g^{-1}\Gamma_o g \cap U(p)) \longrightarrow \mathbb{B}^n(gp,N)/(\Gamma \cap U(gp),$$

$$\varphi^U(\zeta_{g^{-1}\Gamma_o g \cap U(p)}(z)) = \zeta_{\Gamma \cap U(gp)}(gz) \quad \text{for all} \quad z \in \mathbb{B}^n(p,N)$$

and

$$\varphi^W : Z_o(p,N) = \mathbb{B}^n(p,N)/(g^{-1}\Gamma_o g \cap W(p)) \longrightarrow \mathbb{B}^n(gp,N)/(\Gamma \cap W(gp)) = Z(gp,N),$$

$$\varphi^W(\zeta_{g^{-1}\Gamma_o g \cap W(p)}(z)) = \zeta_{\Gamma \cap W(gp)}(gz) \quad \text{for all} \quad z \in \mathbb{B}^n(p,N).$$

There is a trivial extension

$$\varphi^U : (\mathbb{B}^n(p,N)/g^{-1}\Gamma_o g \cap U(p)) \cup (\mathbb{C}^{n-1} \times 0) \longrightarrow (\mathbb{B}^n(gp,N)/\Gamma \cap U(gp)) \cup (\mathbb{C}^{n-1} \times 0),$$

$$\varphi^U(c_1, \ldots, c_{n-1}, 0) = (c_1, \ldots, c_{n-1}, 0) \quad \text{for} \quad \forall (c_1, \ldots, c_{n-1}, 0) \in \mathbb{C}^{n-1} \times 0.$$

Note that the induced homomorphisms

$$\varphi^U_* : g^{-1}\Gamma_o g \cap U(p) \to \Gamma \cap U(gp),$$

$$\varphi^U_*(g^{-1}\gamma_o g) = \gamma_o \quad \text{for} \quad \forall \in \gamma_o \in \Gamma_o$$

of the fundamental groups $g^{-1}\Gamma_o g \cap U(p) = \pi_1(\mathbb{B}^n(p,N)/g^{-1}\Gamma_o g \cap U(p))$, $\Gamma \cap U(gp) = \pi_1(\mathbb{B}^n(gp,N)/\Gamma \cap U(gp))$ and

$$\varphi^W_* : \pi_1(Z_o(p,N)) = g^{-1}\Gamma_o g \cap W(p) \longrightarrow \Gamma \cap W(gp) = \pi_1(Z(gp,N)),$$

$$\varphi^W_*(g^{-1}\gamma_o g) = \gamma_o \quad \text{for} \quad \forall \gamma_o \in \Gamma_o$$

are conjugations by g. On the other hand, $Stab(gp) = gStab(p)g^{-1}$ implies that $W(gp) = gW(p)g^{-1}$, $U(gp) = gU(p)g^{-1}$, so that

$$g\Lambda_o(p)g^{-1} = (\Gamma_o \cap W(gp))/(\Gamma_o \cap U(gp)) \simeq (\Gamma_o \cap W(gp))U(gp)/U(gp)$$

can be viewed as a finite index subgroup of

$$(\Gamma \cap W(p))U(gp)/U(gp) \simeq (\Gamma \cap W(gp))/(\Gamma \cap U(gp)) = \Lambda(gp).$$

Thus, $\varphi^U = \mathrm{Id}_{\mathbb{C}^{n-1}\times 0} : \mathbb{C}^{n-1} \times 0 \to \mathbb{C}^{n-1} \times 0$ induces a homomorphism

$$\varphi^W : T_o(p) = \mathbb{C}^{n-1}/\Lambda_o(p) \longrightarrow \mathbb{C}^{n-1}/\Lambda(gp) = T(gp),$$

$$\varphi^W((c_1,\dots,c_{n-1}) + \Lambda_o(p)) = (c_1,\dots,c_{n-1}) + \Lambda(gp)$$

of the compact complex tori $T_o(p)$, $T(gp)$. The fibres of $\varphi^W : T_o(p) \to T(gp)$ have one and a same cardinality $[\Lambda(gp) : g\Lambda_o(p)g^{-1}]$ over all the points of $T(gp)$, so that the branch index of φ^W is constant over $T_o(p)$. Thus, $\varphi : (\mathbb{B}^n/g^{-1}\Gamma_o g)' \to (\mathbb{B}^n/\Gamma)'$ is a finite covering of the toroidal compactifications of torsion free local complex hyperbolic spaces. Bearing in mind that $\mathbb{P} : SU_{n,1} \to PU_{n,1} = SU_{n,1}/Z(S_{n,1})$ is a group homomorphism with $\mathbb{B}^n/\Gamma_1 = \mathbb{B}^n/\mathbb{P}(\Gamma_1)$ for any lattice Γ_1 of $SU_{n,1}$, one can view φ as a covering $\varphi : (\mathbb{B}^n/\mathbb{P}(g)^{-1}\mathbb{P}(\Gamma_o)\mathbb{P}(g))' \to (\mathbb{B}^n/\mathbb{P}(\Gamma))'$.

(ii) We have already mentioned that $\Gamma_j^U(p) := \Gamma_j \cap U(p)$ are normal subgroups of $\Gamma_j^W(p) := \Gamma_j \cap W(p)$. One checks immediately that the normal subgroup Γ_2 of Γ_1 intersects $W(p)$ in a normal subgroup $\Gamma_2^W(p)$ of $\Gamma_1^W(p)$. The product $\Gamma_1^U(p)\Gamma_2^W(p)$ of the normal subgroups $\Gamma_1^U(p)$, $\Gamma_2^W(p)$ is a normal subgroup of $\Gamma_1^W(p)$ with quotient group

$$\Gamma_1^W(p)/\Gamma_1^U(p)\Gamma_2^W(p) \simeq [\Gamma_1^W(p)/\Gamma_1^U(p)]/[\Gamma_1^U(p)\Gamma_2^W(p)/\Gamma_1^U(p)].$$

By the very construction,

$$\pi_1(T(\zeta_{\Gamma_1}(p))) \simeq \Gamma_1^W(p)/\Gamma_1^U(p)$$

and

$$\Gamma_1^U(p)\Gamma_2^W(p)/\Gamma_1^U(p) \simeq \Gamma_2^W(p)/[\Gamma_2^W(p) \cap \Gamma_1^U(p)] = \Gamma_2^W(p)/\Gamma_2^U(p) \simeq \pi_1(T(\zeta_{\Gamma_2}(p))),$$

so that

$$\Gamma_1^W(p)/\Gamma_1^U(p)\Gamma_2^W(p) \simeq \pi_1 T(\zeta_{\Gamma_1}(p))/\pi_1 T(\zeta_{\Gamma_2}(p))$$

acts effectively on all the fibres of $\varphi : T(\zeta_{\Gamma_2}(p)) \to T(\zeta_{\Gamma_1}(p))$.

(iii) Any finite covering $\varphi : X_2' = (\mathbb{B}^n/\Gamma_2)' \to X_1' = (\mathbb{B}^n/\Gamma_1)'$ of toroidal compactifications of local complex hyperbolic spaces induces a finite unramified covering $\varphi : \mathbb{B}^n/\Gamma_2 \to \mathbb{B}^n/\Gamma_1$. If $\zeta_{\Gamma_j} : \mathbb{B}^n \to \mathbb{B}^n/\Gamma_j$, $1 \leq j \leq 2$ are the universal covering maps of the corresponding ball quotients, then the unramified covering $\varphi \circ \zeta_{\Gamma_2} : \mathbb{B}^n \to \mathbb{B}^n/\Gamma_1$ by the simply connected $\mathbb{B}^n$ lifts across the unramified covering $\zeta_{\Gamma_1} : \mathbb{B}^n \to \mathbb{B}^n/\Gamma_1$ with $\pi_1(\mathbb{B}^n) = 1$ to a holomorphic map $\widetilde{\varphi} : \mathbb{B}^n \to \mathbb{B}^n$, closing the commutative diagram

$$\begin{tikzcd} \mathbb{B}^n \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\zeta_{\Gamma_2}"] & \mathbb{B}^n \arrow[d, "\zeta_{\Gamma_1}"] \\ \mathbb{B}^n/\Gamma_2 \arrow[r, "\varphi"] & \mathbb{B}^n/\Gamma_1 \end{tikzcd} .$$

The map $\widetilde{\varphi}$ is an unramified self covering of the simply connected ball $\mathbb{B}^n$, so that $\widetilde{\varphi}$ is to be a biholomorphism of $\mathbb{B}^n$. In other words, there exists $g \in SU_{n,1}$ with $\widetilde{\varphi}(z) = gz$ for $\forall z \in \mathbb{B}^n$. The fibres of $\zeta_{\Gamma_j} : \mathbb{B}^n \to \mathbb{B}^n/\Gamma_j$ are Γ_j-orbits, $\zeta_{\Gamma_j}^{-1}(\zeta_{\Gamma_j}(z)) = \{\gamma_j z \mid \gamma_j \in \Gamma_j\} = Orb_{\Gamma_j}(z)$. For any fixed $z \in \mathbb{B}^n$ the restriction of g to $g : Orb_{\Gamma_2}(z) \to Orb_{\Gamma_1}(gz)$ is induced by the homomorphism

$$\varphi_* : \pi_1(\mathbb{B}^n/\Gamma_2) = \Gamma_2 \longrightarrow \Gamma_1 = \pi_1(\mathbb{B}^n/\Gamma_1)$$

of the fundamental groups. In other words, $g\gamma_2(z) = \varphi_*(\gamma_2)gz$ for all $\gamma_2 \in \Gamma_2$. Substituting $y = gz$, one obtains $g\gamma_2 g^{-1}y = \varphi_*(\gamma_2)y$ for $\forall y \in \mathbb{B}^n$. Therefore $g\gamma_2 g^{-1}$ and $\varphi_*(\gamma_2) \in \Gamma_1$ from $SU_{n,1}$ define one and a same holomorphic automorphism of $\mathbb{B}^n$. As far as the center $Z(SU_{n,1})$ of $SU_{n,1}$ is the ineffective kernel of the $SU_{n,1}$-action on $\mathbb{B}^n$, one concludes that $\mathbb{P}(g\Gamma_2 g^{-1})$ is a subgroup of $\mathbb{P}(\Gamma_1)$. The considerations at the beginning of the proof of (i) establish that $g : \mathbb{B}^n \to \mathbb{B}^n$ induces an isomorphism

$$\psi : \mathbb{B}^n/\mathbb{P}(\Gamma_2) = \mathbb{B}^n/\Gamma_2 \longrightarrow \mathbb{B}^n/(g^{-1}\Gamma_2 g^{-1}) = \mathbb{B}^n/\mathbb{P}(g\Gamma_2 g^{-1}).$$

That provides a commutative diagram

$$\begin{tikzcd} \mathbb{B}^n/\mathbb{P}(\Gamma_2) \arrow[r, "\psi"] \arrow[d, "\varphi"'] & \mathbb{B}^n/\mathbb{P}(g\Gamma_2 g^{-1}) \arrow[dl, "\varphi\circ\psi^{-1}"] \\ \mathbb{B}^n/\mathbb{P}(\Gamma_1) & \end{tikzcd}$$

with finite unramified covering $\varphi\circ\psi^{-1}$, induced by the inclusion $\mathbb{P}(g\Gamma_2 g^{-1}) \hookrightarrow \mathbb{P}(\Gamma_1)$. In particular, $[\mathbb{P}(\Gamma_1) : \mathbb{P}(g\Gamma_2 g^{-1})] = \deg(\varphi \circ \psi^{-1}) = \deg(\varphi)$.

$\square$

The results of McReynolds from [22] imply that for any arithmetic lattice $\Gamma < SU_{n,1}$, any Γ-rational boundary point $p \in \partial_\Gamma \mathbb{B}^n$ and any $g \in \Gamma \setminus Stab(p)$ there is a subgroup $\Gamma_o < \Gamma$ of finite index $[\Gamma : \Gamma_o] < \infty$ with $\Gamma(p) := \Gamma \cap Stab(p) \leq \Gamma_o$ and $g \notin \Gamma_o$. By Lemma 9 (i), the Γ_o-rational boundary points $\partial_{\Gamma_o}\mathbb{B}^n = \partial_\Gamma \mathbb{B}^n$ coincide with the Γ-rational ones. The coincidence $\Gamma_o(p) = \Gamma(p)$ of the stabilizers of $p \in \partial_\Gamma \mathbb{B}^n$ in Γ_o and Γ requires the presence of at least two Γ_o-cusps over $\kappa = \zeta_\Gamma(p) \in \partial_\Gamma \mathbb{B}^n/\Gamma$. Otherwise, there follows $Orb_\Gamma(p) = Orb_{\Gamma_o}(p)$ and for any $\gamma \in \Gamma$ there exists $\gamma_o \in \Gamma_o$ with $\gamma(p) = \gamma_o(p)$. As a result, $\gamma_o^{-1}\gamma \in \Gamma \cap Stab(p) = \Gamma_o(p) < \Gamma_o$, whereas $\gamma \in \Gamma_o$ and $\Gamma = \Gamma_o$. The contradiction justifies that for any arithmetic quotient $\mathbb{B}^n/\Gamma$ and any Γ-cusp $\kappa \in \partial_\Gamma \mathbb{B}^n/\Gamma$ there exists a finite covering

$$\widehat{\mathbb{B}^n/\Gamma_o} = (\mathbb{B}^n/\Gamma_o) \cup (\partial_\Gamma \mathbb{B}^n/\Gamma_o) \longrightarrow \widehat{\mathbb{B}^n/\Gamma} = (\mathbb{B}^n/\Gamma) \cup (\partial_\Gamma \mathbb{B}^n/\Gamma)$$

of the corresponding Baily-Borel compactifications $\widehat{\mathbb{B}^n/\Gamma_o}$, $\widehat{\mathbb{B}^n/\Gamma}$, which is not totally ramified over κ.

If $f : X_2' = (\mathbb{B}^n/\Gamma_2)' \to (\mathbb{B}^n/\Gamma_1)' = X_1'$ is a finite covering of toroidal compactifications of torsion free local complex hyperbolic spaces $\mathbb{B}^n/\Gamma_j$ and Γ_2 is a normal

subgroup of Γ_1, then f is Γ_1/Γ_2-Galois covering. For any irreducible component T_i of $T = (\mathbb{B}^n/\Gamma_1)' \setminus (\mathbb{B}^n/\Gamma_1)$ and any irreducible component $T_{i,j}$ of $f^{-1}(T_i)$ the restriction $f_{i,j} : T_{i,j} \to T_i$ is a finite Galois covering of compact complex $(n-1)$-dimensional tori. The stabilizers of all points $q \in T_{i,j}$ in Γ_1/Γ_2 have one and a same order, which is called the ramification index $\rho_{i,j}$ of f over $T_{i,j}$. The degree of $f_{i,j} : T_{i,j} \to T_i$ equals $\deg(f_{i,j}) = \frac{\deg(f)}{\rho_{i,j}}$.

Corollary 10. *Let $X' = (\mathbb{B}^n/\Gamma)'$ be the toroidal compactification of a torsion free local complex hyperbolic space $\mathbb{B}^n/\Gamma$ and $T = (\mathbb{B}^n/\Gamma)' \setminus (\mathbb{B}^n/\Gamma)$ be the toroidal compactifying divisor of $\mathbb{B}^n/\Gamma$. Then:*

(i) for any irreducible component T_i of T and any $N \in \mathbb{N}$ there is a finite Galois covering

$$\varphi_{i,N} : X'_{i,N} = (\mathbb{B}^n/\Gamma_{i,N})' \longrightarrow X' = (\mathbb{B}^n/\Gamma)'$$

of torsion free toroidal compactifications with ramification index $> N$ over any irreducible component of $\varphi_{i,N}^{-1}(T_i)$.

(ii) for any $N \in \mathbb{N}$ there is a finite Galois covering

$$\varphi_N : X'_N = (\mathbb{B}^n/\Gamma_N)' \longrightarrow X' = (\mathbb{B}^n/\Gamma)'$$

of torsion free toroidal compactifications with ramification index $> N$ over any irreducible component of $T(N) = (\mathbb{B}^n/\Gamma_N)' \setminus (\mathbb{B}^n/\Gamma_N)$.

Proof. (i) Let $\kappa \in \partial_\Gamma \mathbb{B}^n/\Gamma$ be the corresponding cusp of T_i and $p \in \partial_\Gamma \mathbb{B}^n$ be a Γ-rational boundary point over $\kappa = \zeta_\Gamma(p)$. The commutant $U(p)$ of the unipotent radical $W(p)$ of $Stab(p) < SU_{n,1}$ intersects Γ in an infinite cyclic group $\Gamma^U(p) := \Gamma \cap U(p) \simeq (\mathbb{Z}, +)$. Choose a generator $c(p)$ of $\Gamma^U(p)$ and note that $c(p)^{N!} \neq \mathrm{Id}_{\mathbb{B}^n}$. By the residual finiteness of the lattice Γ of $SU_{n,1}$, there is a normal subgroup $\Gamma_{i,N} \triangleleft \Gamma$ of finite index $[\Gamma : \Gamma_{i,N}] < \infty$, such that $c(p)^{N!} \not\in \Gamma_{i,N}$. For any $\gamma \in \Gamma$ recall that $Stab(\gamma p) = \gamma Stab(p) \gamma^{-1}$, $W(\gamma p) = \gamma W(p) \gamma^{-1}$, $U(\gamma p) = \gamma U(p) \gamma^{-1}$, so that $c(\gamma p) := \gamma c(p) \gamma^{-1}$ is a generator of $\Gamma^U(\gamma p) = \Gamma \cap U(\gamma p)$. Note that $c(\gamma p)^{N!} = \gamma c(p)^{N!} \gamma^{-1} \not\in \Gamma_{i,N}$, as far as $\Gamma_{i,N}$ is a normal subgroup of Γ. Thus, for any $q \in \zeta_\Gamma^{-1}(\kappa)$ one has a generator $c(q)$ of $\Gamma^U(q)$ with $c(q)^{N!} \not\in \Gamma_{i,N}$. The coset $c(q)\Gamma_{i,N}$ is of order $o(q) > N$ in the quotient group $\Gamma/\Gamma_{i,N}$, since otherwise $c(q)^{o(q)} \in \Gamma_{i,N}$ requires $c(q)^{N!} \in \Gamma_{i,N}$. Let $\varphi_{i,N} : X'_{i,N} = (\mathbb{B}^n/\Gamma_{i,N})' \to X' = (\mathbb{B}^n/\Gamma)'$ be the finite $\Gamma/\Gamma_{i,N}$-Galois covering, induced by $\mathrm{Id}_{\mathbb{B}^n} : \mathbb{B}^n \to \mathbb{B}^n$. According to Lemma 9 (ii), all the fibres of the restriction

$$\varphi_{i,N} : T(\zeta_{\Gamma_{i,N}}(q)) \longrightarrow T(\zeta_\Gamma(q)) = T(\kappa) = T_i$$

are acted effectively by the quotient group $\Gamma^W(q)/\Gamma^U(q)\Gamma^W_{i,N}(q)$. Therefore there are at least $N+1$ different elements $c(q)^m \Gamma_{i,N} \in \Gamma/\Gamma_{i,N}$, $0 \leq m \leq N < N+1 \leq o(q)$, whose liftings $c(q)^m \in \Gamma^U(q) \subset \Gamma^U(q)\Gamma^W_{i,N}$ stabilize all the points of $T(\zeta_{\Gamma_{i,N}}(q))$. Any irreducible component of $\varphi_{i,N}^{-1}(T_i)$ is of the form $T(\zeta_{\Gamma_{i,N}}(q))$ for some $q \in \zeta_\Gamma^{-1}(\kappa)$.

(ii) Suppose that $T = \sum\limits_{i=1}^{h} T_i$ has h irreducible components. For any $1 \leq i \leq h$ let us choose a normal subgroup $\Gamma_{i,N} \triangleleft \Gamma$ of finite index $[\Gamma : \Gamma_{i,N}] < \infty$, such that

the $\Gamma/\Gamma_{i,N}$-Galois covering $\varphi_{i,N} : X'_{i,N} = (\mathbb{B}^n/\Gamma_{i,N})' \to X' = (\mathbb{B}^n/\Gamma)'$, induced by $\mathrm{Id}_{\mathbb{B}^n} : \mathbb{B}^n \to \mathbb{B}^n$ has ramification index $> N$ over any irreducible component of $\varphi_{i,N}^{-1}(T_i)$. The intersection

$$\Gamma_N := \Gamma_{1,N} \cap \Gamma_{2,N} \cap \ldots \cap \Gamma_{h,N}$$

is a normal subgroup of Γ. More precisely, by an induction on $1 \leq i \leq N$, we check that $\Gamma_N^{(i)} := \Gamma_{1,N} \cap \ldots \cap \Gamma_{i,N}$ is of finite index in Γ, as far as

$$\Gamma_N^{(i-1)}/\Gamma_N^{(i)} = \Gamma_N^{(i-1)}/\Gamma_N^{(i-1)} \cap \Gamma_{i,N} \simeq \Gamma_N^{(i-1)}\Gamma_{i,N}/\Gamma_{i,N}$$

is a subgroup of the finite group $\Gamma/\Gamma_{i,N}$ and the index

$$[\Gamma : \Gamma_N^{(i)}] = [\Gamma : \Gamma_N^{(i-1)}][\Gamma_N^{(i-1)} : \Gamma_N^{(i)}].$$

The Γ/Γ_N-Galois covering $\varphi_N : X'_N = (\mathbb{B}^n/\Gamma_N)' \to X' = (\mathbb{B}^n/\Gamma)'$, induced by $\mathrm{Id}_{\mathbb{B}^n} : \mathbb{B}^n \to \mathbb{B}^n$ has a factorization

$$X'_N = (\mathbb{B}^n/\Gamma_N)' \xrightarrow{\rho_h} (\mathbb{B}^n/\Gamma_N^{(h-1)})' \xrightarrow{\rho_{h-1}} (\mathbb{B}^n/\Gamma_N^{(h-2)})' \xrightarrow{\rho_{h-2}} \cdots$$

$$\cdots \xrightarrow{\rho_3} (\mathbb{B}^n/\Gamma_N^{(2)})' \xrightarrow{\rho_2} (\mathbb{B}^n/\Gamma_{1,N})' \xrightarrow{\rho_1} X' = (\mathbb{B}^n/\Gamma)'$$

into a product of $\Gamma_N^{(i)}/\Gamma_N^{(i-1)}$-Galois coverings $\rho_i : (\mathbb{B}^n/\Gamma_N^{(i)})' \to (\mathbb{B}^n/\Gamma_N^{(i-1)})'$ for $1 \leq i \leq h$. By an induction on i, if

$$\psi_i := \rho_1 \ldots \rho_{i-1}\rho_i : (\mathbb{B}^n/\Gamma_N^{(i)})' \longrightarrow X' = (\mathbb{B}^n/\Gamma)'$$

has ramification index $> N$ over the irreducible components of $\psi_i^{-1}(T_j)$, $\forall 1 \leq j \leq i$, then

$$\psi_{i+1} = \psi_i \circ \rho_{i+1} : (\mathbb{B}^n/\Gamma_N^{(i+1)})' \longrightarrow X' = (\mathbb{B}^n/\Gamma)'$$

has ramification index $> N$ over the irreducible components of $\psi_{i+1}^{-1}(T_j)$ for all $1 \leq j \leq i$. There is a factorization

$$\begin{tikzcd}
(\mathbb{B}^n/\Gamma_N^{(i+1)})' \arrow[r, "\psi_{i+1}"] \arrow[d] & X' = (\mathbb{B}^n/\Gamma)') \\
(\mathbb{B}^n/\Gamma_{i+1,N})' \arrow[ur, "\varphi_{i+1,N}"'] &
\end{tikzcd}$$

into a product of finite Galois coverings, which reveals that the ramification index of ψ_{i+1} is $> N$ over the irreducible components of $\psi_{i+1}^{-1}(T_{i+1})$. By the very definition, $\psi_h = \varphi_N$.

$\square$

Ash-Mumford-Rapoport-Tsai proved in [1] that for any arithmetic lattice Γ of the biholomorphism group G of a bounded symmetric domain $D = G/K$ there is a subgroup $\Gamma_o < \Gamma$ of finite index, such that D/Γ_o is a variety of general type. An arbitrary finite covering $\varphi : X_2' = (\mathbb{B}^n/\Gamma_2)' \to X_1' = (\mathbb{B}^n/\Gamma_1)'$ of torsion free toroidal compactifications relates the canonical divisors $K_{X_j'}$ of X_j' by the formula $K_{X_2'} = \varphi^* K_{X_1'} + R(\varphi)$, where $R(\varphi)$ stands for the ramification locus of φ. Thus, Corollary 10 (ii) may be viewed as a numerical specification of the aforementioned Ash-Mumford-Rapoport-Tsai's result.

Acknowledgements: The author is extremely grateful to the referee for pointing out a flow in the description of the torsion free toroidal compactification , as well as for bringing to her attention McReynolds' [22] and Stover's [24].

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